

AMBIKÄHLER GEOMETRY, AMBITORIC SURFACES AND EINSTEIN 4-ORBIFOLDS

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ABSTRACT. We give an explicit local classification of conformally equivalent but oppositely oriented Kähler metrics on a 4-manifold which are toric with respect to a common 2-torus action. In the generic case, these structures have an intriguing local geometry depending on a quadratic polynomial and two arbitrary functions of one variable, these two functions being explicit degree 4 polynomials when the Kähler metrics are extremal (in the sense of Calabi).

One motivation for and application of this result is an explicit local description of Einstein 4-manifolds which are hermitian with respect to either orientation. This can be considered as a riemannian analogue of a result in General Relativity due to R. Debever, N. Kamran, and R. McLenaghan, and is a natural extension of the classification of selfdual Einstein hermitian 4-manifolds, obtained independently by R. Bryant and the first and third authors.

We discuss toric compactifications of these metrics on orbifolds and provide infinite discrete families of compact toric extremal Kähler orbifolds. Our examples include Bach-flat Kähler orbifolds which are conformal to complete smooth Einstein metrics on an open subset. We illustrate how these examples fit with recent conjectures relating the existence of extremal toric metrics to various notions of stability.

1. INTRODUCTION

This paper concerns pairs of Kähler metrics in real dimension four which induce the same conformal structure with opposite orientations, and are also toric with respect to a common 2-torus action. More precisely, we consider a 4-manifold (or 4-orbifold) M equipped with Kähler metrics (g_+, J_+, ω_+) and (g_-, J_-, ω_-) such that

- g_+ and g_- belong to the same conformal class (i.e., $g_- = f^2 g_+$ for a positive function f on M);
- J_+ and J_- induce opposite orientations (i.e., the volume elements $\frac{1}{2}\omega_+ \wedge \omega_+$ and $\frac{1}{2}\omega_- \wedge \omega_-$ on M have opposite signs);
- there is a 2-dimensional subspace of vector fields on M , linearly independent on a dense open set, whose elements are hamiltonian and Poisson-commuting Killing vector fields with respect to both (g_+, ω_+) and (g_-, ω_-) .¹

We refer to such data as an *ambitoric structure* on M .

Our motivation for studying ambitoric geometry is twofold. The first comes from the theory of hamiltonian 2-forms in four dimensions [6]. Indeed, any orthotoric Kähler metric and certain Kähler metrics of Calabi type on a 4-manifold admit a conformally equivalent Kähler metric and two commuting Killing fields satisfying the above conditions. We recall these constructions in sections 4.1 and 4.2 below. There is also a link between ambitoric conformal metrics and symmetric Killing $(2, 0)$ -tensors, which we explain in an appendix.

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¹If ω is a symplectic form, hamiltonian vector fields $K_1 = \text{grad}_\omega f_1$ and $K_2 = \text{grad}_\omega f_2$ Poisson-commute iff the Poisson bracket $\{f_1, f_2\}$ with respect to ω is zero. This holds iff $\omega(K_1, K_2) = 0$.

The second class of examples that naturally involve ambitoric structures comes from the theory of Einstein 4-manifolds (M, g) with algebraically special half-Weyl tensors W^\pm . By this we mean that at any point of M at least two of the three eigenvalues of W^\pm coincide, when W^+ and W^- are viewed as symmetric tracefree operators acting on the three-dimensional spaces of selfdual and antiselfdual 2-forms respectively.² The riemannian Goldberg–Sachs theorem [4] and the work of Derdziński [22] imply that such Einstein metrics admit (generically unique up to signs) orthogonal complex structures J_+ and J_- which are conformally Kähler. Moreover, in the case when W^\pm do not vanish, the conformal Kähler metrics are given by $g_\pm = |W^\pm|_g^{2/3} g$, and conversely $g = s_\pm^{-2} g_\pm$, where s_\pm are the scalar curvatures of g_\pm . From the J_\pm -invariance of the Ricci tensor of g , it then follows that $\text{grad}_{\omega_\pm} s_\pm$ are commuting Killing vector fields for g_\pm , yielding an ambitoric structure unless they are linearly dependent. The special case of selfdual Einstein 4-manifolds with algebraically special selfdual Weyl tensor has been already worked out in [5]. With a little more work, we prove in Theorem 1 that *any* Einstein 4-manifold with algebraically special half-Weyl tensors locally admits a compatible ambitoric structure.

Thus motivated, we study ambitoric structures in general: we show that in a neighbourhood of any point, they are either of Calabi type (and hence classified by well-known results), or “*regular*”. We then obtain, in Theorem 2, an explicit local classification of regular ambitoric structures in terms of a quadratic polynomial q and two arbitrary functions A and B of one variable which are positive on some interval. The classification reveals a subtle geometry behind regular ambitoric structures which we attempt to elucidate, but some features remain mysterious.

For practical purposes, however, our classification reduces curvature conditions on ambitoric structures to systems of functional ODE’s, rather than PDE’s, and we explore this in greater detail in section 6, where we compute the Ricci forms and scalar curvatures for an arbitrary regular ambitoric pair (g_+, g_-) of Kähler metrics. By solving the resulting equations, we then prove, in Theorem 3, that g_+ is extremal (in the sense of Calabi) if and only if g_- is; further A and B are then explicitly determined as polynomials of degree at most 4 whose coefficients satisfy three independent linear equations depending on q , which we solve for 7 free parameters (one of which can be seen as a homothety factor of the metric). Among these structures, those containing an Einstein metric in their conformal class are characterized by the vanishing of the Bach tensor (see section 3.2) which in turn places one further quadratic relation on the coefficients of A and B . This yields an explicit local description of all Einstein 4-dimensional manifolds (M, g) with algebraically special half-Weyl tensors (Corollary 1).

The explicit form of extremal ambitoric structures makes it possible to study their compactifications as toric Kähler orbifolds using rational Delzant polytopes (see [21, 38]) defined by the roots of the polynomials A and B appearing in the local classification. We use the description of compact toric Kähler orbifolds given by Guillemin [30] and Abreu [2] in order to obtain sufficient conditions for an ambitoric metric to compactify (Proposition 11). This leads to infinite discrete families of extremal Kähler orbifolds. Among them are Bach-flat Kähler orbifolds which are globally conformally Einstein as well as examples where there is an open set on which the Kähler metric is conformal to a smooth, complete Einstein metric.

²This class of riemannian 4-manifolds can be seen as an analogue of the Einstein lorentzian 4-manifolds with Weyl tensor of Petrov type D , which has been extensively studied in General Relativity, and classified by R. Debever, N. Kamran and R. G. McLenaghan in [20].

In section 9 we illustrate how these toric orbifolds fit in into the conjectures by Donaldson [23] and Székelyhidi [43], relating the existence of extremal toric metrics to various notions of stability of the underlying varieties. In our specific case, it turns out that the notion of K-polystability with respect to toric degenerations introduced in [23] is directly related to the positivity of the two polynomials $A(z)$ and $B(z)$ appearing in the definition of the extremal ambitoric metrics. We use this observation to define a special class of rational Delzant polytopes in \mathbb{R}^2 for which Donaldson's conjecture holds true (Theorem 4). This yields explicit examples of (unstable) toric orbifolds which do not admit extremal Kähler metrics.

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2. AMBIHERMITIAN AND AMBIKÄHLER GEOMETRY

2.1. Conformal hermitian structures. Let M be a 4-dimensional manifold. A *hermitian metric* on M is defined by a pair (g, J) consisting of a riemannian metric $g \in C^\infty(M, S^2 T^* M)$ and an integrable almost complex structure $J \in C^\infty(M, \text{End}(TM))$, which are *compatible* in the sense that $g(J\cdot, J\cdot) = g(\cdot, \cdot)$.

The *fundamental 2-form* or *Kähler form* of (g, J) is defined by $\omega^g(\cdot, \cdot) := g(J\cdot, \cdot)$; it is a J -invariant 2-form of square-norm 2. The volume form $v_g = \frac{1}{2}\omega^g \wedge \omega^g$ induces an orientation on M (coinciding with the complex orientation of J) with respect to which ω^g is a section of the bundle $\wedge^+ M$ of *selfdual* 2-forms; the bundle $\wedge^- M$ of *antiselfdual* 2-forms is then identified with the bundle of J -invariant 2-forms orthogonal to ω^g .

For any metric $\tilde{g} = f^{-2}g$ conformal to g (where f is a positive function on M), the pair (\tilde{g}, J) is also hermitian. The *Lee 1-form*, θ^g , of (g, J) is introduced by

$$(1) \quad d\omega^g = -2\theta^g \wedge \omega^g,$$

or equivalently $\theta^g = -\frac{1}{2}J\delta^g\omega^g$, where δ^g is the co-differential with respect to the Levi-Civita connection D^g of g . Since J is integrable, $d\omega^g$ measures the deviation of (g, J) from being a Kähler structure (for which J and ω^g are parallel with respect to D^g). Thus a hermitian 4-manifold is Kähler if and only if $\theta^g = 0$. Indeed $D^g\omega^g$ is given by the equation

$$(2) \quad D_X^g\omega^g = J\theta^g \wedge X^\flat + \theta^g \wedge JX^\flat,$$

where $X^\flat := g(X, \cdot)$ denotes the 1-form dual to the vector field X (see e.g., [4]).

For conformally equivalent metrics $\tilde{g} = f^{-2}g$, the corresponding Lee forms are linked by $\theta^{\tilde{g}} = \theta^g + d\log f$; it follows that there is a Kähler metric in the conformal class if and only if θ^g is exact; locally, this is true if and only if $d\theta^g = 0$ and g is uniquely determined up to homothety.

Remark 1. A conformally invariant (and well known) interpretation of the Lee form may be obtained from the observation that a conformal class of riemannian metrics determines and is determined by an oriented line subbundle of $S^2 T^* M$ whose positive sections are the riemannian metrics in the conformal class. Writing

this line subbundle as $\Lambda^2 := \Lambda \otimes \Lambda$ (with Λ also oriented), it is thus equivalently a bundle metric c on $\Lambda \otimes TM$ and the volume form of this bundle metric identifies Λ^4 with $\wedge^4 T^*M$. A metric in the conformal class may be written $g = \ell^{-2}c$ for a positive section ℓ of the line bundle $L = \Lambda^*$; such an ℓ is called a *length scale*.

Any connection on TM induces a connection on $L = (\wedge^4 TM)^{1/4}$; in particular the Levi-Civita connection D^g of $g = \ell^{-2}c$ induces the unique connection (also denoted D^g) on L with $D^g \ell = 0$. More generally, a connection D on TM is said to be *conformal* if $Dc = 0$. It is well known (see e.g. [15]) that taking the induced connection on L defines an affine bijection from the affine space of torsion-free conformal connections on TM (the *Weyl connections*) to the affine space of connections on L (modelled on the vector space of 1-forms).

If J is hermitian with respect to c , it is easy to verify that the connection $D^g + \theta^g$ on L is independent of the choice of metric $g = \ell^{-2}c$ in the conformal class. Equation (2) then has the interpretation that the induced torsion-free conformal connection D^J on M is the unique such connection with $D^J J = 0$, while $d\theta^g$ may be viewed as the curvature of the corresponding connection on L .

In view of this remark, we will find it more natural in this paper to view a hermitian structure as a pair (c, J) where c is a *conformal metric* as above, and J is a complex structure which is orthogonal with respect to c (i.e., $c(J\cdot, J\cdot) = c(\cdot, \cdot)$). We refer to (M, c, J) as a hermitian complex surface. A compatible hermitian metric is then given by a metric $g = \ell^{-2}c$ in the corresponding conformal class.

2.2. Ambihhermitian and ambikähler structures.

Definition 1. Let M be a 4-manifold. An *ambihhermitian structure* is a triple (c, J_+, J_-) consisting of a conformal metric c and two c -orthogonal complex structures J_\pm such that J_+ and J_- induce opposite orientations on M .³

A metric $g = \ell^{-2}c$ in the conformal class is called an ambihhermitian metric on (M, J_+, J_-) and we denote by ω_\pm^g (resp. θ_\pm^g) the fundamental 2-forms (resp. the Lee forms) of the hermitian metrics (g, J_\pm) .

The following elementary and well-known observation will be used throughout.

Lemma 1. *Let M be a 4-manifold endowed with a pair (J_+, J_-) of almost complex structures inducing different orientations on M . Then M admits a conformal metric c for which both J_+ and J_- are orthogonal if and only if J_+ and J_- commute. In this case, the tangent bundle TM splits as a c -orthogonal direct sum*

$$(3) \quad TM = T_+M \oplus T_-M$$

of J_\pm -invariant rank 2 subbundles $T_\pm M$ defined as the ± 1 -eigenbundles of $-J_+J_-$.

It follows that an ambihhermitian metric is equivalently given by a pair of commuting complex structures on M and hermitian metrics on each of the complex line subbundles T_+M and T_-M .

Definition 2. An ambihhermitian conformal 4-manifold (M, c, J_+, J_-) is called *ambikähler* if it admits ambihhermitian metrics g_+ and g_- such that (g_+, J_+) and (g_-, J_-) are Kähler metrics. (It follows that an ambihhermitian structure is ambikähler if and only if with respect to any ambihhermitian metric $g = \ell^{-2}c$, the corresponding Lee forms θ_\pm^g are exact.)

³The prefix *ambi-* means “on both sides”, often left and right: ambihhermitian structures have complex structures of either handedness (orientation); they should be contrasted (and not confused) with *bihhermitian structures* where J_\pm induce the *same* orientation on M .

With slight abuse of notation, we denote henceforth by ω_+ and ω_- the corresponding (symplectic) Kähler forms, thus omitting the upper indices indicating the corresponding Kähler metrics g_+ and g_- . Similarly we set $v_{\pm} = \frac{1}{2}\omega_{\pm} \wedge \omega_{\pm}$.

3. EINSTEIN METRICS AND AMBIKÄHLER 4-MANIFOLDS

3.1. Algebraically special Einstein 4-manifolds. One motivation for studying ambikähler 4-manifolds is the classification of Einstein 4-manifolds with algebraically special half-Weyl tensors.

The riemannian Goldberg–Sachs theorem [4] and the work of Derdziński [22] imply that there is a one-to-one correspondence between oriented Einstein riemannian 4-manifolds admitting, about each point, a compatible ambihermitian pair (J_+, J_-) and oriented Einstein riemannian 4-manifolds for which both half-Weyl tensors, W^+ and W^- , are everywhere *degenerate* in the sense that at any point of M at least two of the three eigenvalues of W^+ (resp. W^-) coincide, where W^+ (resp. W^-) is viewed as a symmetric tracefree operator acting on the three-dimensional space of selfdual (resp. antiselfdual) 2-forms. Furthermore, we know from [22] that on each connected component of M each of W^+ and W^- either vanishes identically, or has no zero, in which case it has exactly two distinct eigenvalues (one simple and one of multiplicity 2) on that component.

If W^{\pm} both vanish, then g is a real space form, i.e., is locally isometric to S^4 , \mathbb{R}^4 or H^4 , hence trivially ambikähler. If instead the Einstein metric g is half conformally-flat but not conformally-flat, we can assume $W^- = 0$, $W^+ \neq 0$, by changing the orientation of M if necessary. Then, W^+ is degenerate if and only if g is an selfdual Einstein hermitian metric (see [5] for a classification). In either case, the underlying conformal structure of the Einstein metric is ambikähler with respect to some hermitian structures J_{\pm} (see also the proof of Theorem 1 below).

Therefore, let us assume that W^+ and W^- are both nonvanishing. Then, according to [4, 22], the normalized generators of the simple eigenspaces of W^{\pm} are Kähler forms of (unique up to sign) compatible hermitian structures J_{\pm} ; moreover, the conformally equivalent metrics $g_{\pm} = |W^{\pm}|_g^{2/3} g$ are Kähler with respect to J_{\pm} (respectively). Thus, we obtain a canonically defined ambikähler structure. The following proposition summarizes the situation.

Proposition 1. [4, 22] *For an oriented conformal 4-manifold (M, c) with a compatible Einstein metric $g = \ell^{-2}c$, the following three conditions are equivalent:*

- *both half-Weyl tensors W^+ and W^- are degenerate;*
- *about each point of M there exists a pair of complex structures J_+ and J_- such that (c, J_+, J_-) is ambihermitian;*
- *about each point M there exists a pair of complex structures J_+ and J_- such that (c, J_+, J_-) is ambikähler.*

If M is simply connected and W^{\pm} are both nonzero, then the compatible ambikähler structure (J_+, J_-) is unique (up to signs of J_{\pm}) and globally defined.

3.2. Bach-flat ambikähler conformal structures. In order to characterize those ambikähler structures which are conformally Einstein per Proposition 1, we recall some properties of the *Bach tensor* B of a conformal 4-manifold, which may be characterized as the gradient of the functional $c \mapsto \int_M |W|_c^2$ under compactly supported variations of the conformal metric c , where W is the Weyl tensor of c .

B is a co-closed tracefree section of $L^{-2} \otimes S^2 T^*M$. For any riemannian metric $g = \ell^{-2}c$ in the conformal class, $B^g = \ell^2 B$ is a symmetric bilinear form on TM

defined by the well-known expressions [11, 6]

$$(4) \quad B^g = \delta^g \delta^g W + W_g(S^g) = 2(\delta^g \delta^g W^\pm + (W^\pm)_g(S^g)),$$

where S^g is the *normalized Ricci tensor* of g , defined by $S^g = \frac{1}{2}(r_g - \frac{1}{6}s_g g)$ with r_g and s_g being the usual Ricci tensor and scalar curvature of g respectively. In these formulae we use the action of Weyl tensors on symmetric bilinear forms h by $W_g(h)(X, Y) = \sum_{i=1}^4 g(W_{X, e_i} Y, h^g(e_i))$ where $\{e_i\}$ is any orthonormal frame, and h^g stands for the corresponding endomorphism determined by g . We adopt similar definition for the actions of W^\pm . It immediately follows from these expressions that c is Bach-flat (i.e., B is identically zero) if W^+ or W^- is identically zero.

The conformal invariance of B implies that $B^{f^{-2}g} = f^2 B^g$, and using the second Bianchi identity one sees that

$$(\delta^g W)(X)_{Y,Z} = -(D_Y^g S^g)_{Z,X} + (D_Z^g S^g)_{Y,X}.$$

Thus c is also Bach-flat if the conformal class contains an Einstein metric.

If J is a complex structure compatible with the chosen orientation and \hat{g} is Kähler with respect to J , then $W^+ = \frac{1}{8}s_{\hat{g}}(\omega^{\hat{g}} \otimes \omega^{\hat{g}})_0$ (where $(\cdot)_0$ denotes tracefree part), and the Bach tensor is easily computed by using (4): if $B^{\hat{g},+}$ and $B^{\hat{g},-}$ denote the J -invariant and J -anti-invariant parts of $B^{\hat{g}}$, respectively, then (see [22])

$$(5) \quad B^{\hat{g},+} = \frac{1}{6}(2D^+ ds_{\hat{g}} + s_{\hat{g}} r_{\hat{g}})_0, \quad B^{\hat{g},-} = -\frac{1}{6}D^- ds_{\hat{g}},$$

where, for any real function f , $D^+ df$, resp. $D^- df$, denotes the J -invariant part, resp. the J -anti-invariant part, of the Hessian $D^{\hat{g}} df$ of f with respect to \hat{g} , and h_0 denotes the tracefree part of a bilinear 2-form h .

It follows [22] that the Kähler metric (\hat{g}, J) is *extremal* (i.e., $J \text{grad}_{\hat{g}} s_{\hat{g}}$ is a Killing vector field) if and only if $B^{\hat{g}}$ is J -invariant. In this case the Ricci tensor of the metric $g := s_{\hat{g}}^{-2} \hat{g}$ (defined wherever $s_{\hat{g}}$ is nonzero) is J -invariant. Further, $B^{\hat{g}}$ is identically zero if and only if $g = s_{\hat{g}}^{-2} \hat{g}$ is an Einstein metric wherever it is defined.

Applying this result to either Kähler metric of an ambikähler conformal structure (c, J_+, J_-) we obtain a useful characterization of the ambikähler conformal classes arising from Proposition 1.

Proposition 2. *Let (M, c, J_+, J_-) be a connected ambikähler 4-manifold. Then c is Bach-flat if and only if there is a compatible Einstein metric $g = \ell^{-2}c$ defined on a dense open subset of M .*

Proof. If c is Bach-flat then both of the Kähler metrics (g_+, J_+) and (g_-, J_-) are extremal, so their scalar curvatures, s_+ and s_- , are Killing potentials. By the unique continuation principle, each of s_{\pm} is either nonvanishing on an open dense subset of M or is identically zero. In the former case, at least one of the conformal metrics $s_+^{-2}g_+$ and $s_-^{-2}g_-$ is well-defined and Einstein on an open dense subset of M . On the other hand, if s_{\pm} are both identically zero, so are W^\pm , i.e., c is a flat conformal structure. In particular, there are compatible Einstein metrics on any simply connected open subset of M .

Conversely if there is compatible Einstein metric on a dense open subset, then, as already noted, B vanishes identically there, hence everywhere by continuity. \square

We end this section with the following observation.

Lemma 2. *Let (M, c, J_+, J_-) be a connected ambikähler conformal 4-manifold which is not conformally-flat and for which the corresponding Kähler metrics g_+ ,*

g_- are extremal, but not homothetic. Then c is Bach-flat if and only if the scalar curvatures s_\pm of g_\pm are related by

$$(6) \quad C_+ s_- = C_- \left(\frac{-v_-}{v_+} \right)^{1/4} s_+,$$

where C_\pm are constants not both zero and v_\pm are the volume forms of (g_\pm, J_\pm) .

Proof. If s_+ or s_- is identically zero, (M, c) is half-conformally-flat and (with C_+ or C_- zero) the result is trivial. Otherwise, if c is Bach-flat, $s_+^{-2}g_+$ and $s_-^{-2}g_-$ are Einstein metrics defined on open sets with dense intersection, so they must be homothetic, since (M, c) is not conformally-flat. Thus

$$(7) \quad s_+^{-2}g_+ = C s_-^{-2}g_-,$$

for a positive real number C , and (6) holds with $(C_-/C_+)^2 = C$.

Conversely, with s_\pm nonzero, (6) implies (7), and we may assume g_\pm are chosen so that $g := s_+^{-2}g_+ = s_-^{-2}g_-$ (i.e., $C = 1$). Applying a well known result (see [22, 4]) to both g_+ and g_- , we deduce that W^+ and W^- are both harmonic with respect to g : $\delta^g W^+ = \delta^g W^- = 0$. Thus from (4),

$$(8) \quad B^g = 2(W^\pm)_g(S^g).$$

Moreover, since (g_+, J_+) and (g_-, J_-) are both extremal by assumption, S^g is J_+ - and J_- -invariant. It follows that $S^g = \frac{1}{24}s_g g + \frac{1}{2}\kappa(J_+J_-)_g$, where $(J_+J_-)_g$ denotes the symmetric bilinear form defined by

$$(J_+J_-)_g(X, Y) = g(J_+J_-X, Y) = -g(J_+X, J_-Y),$$

and κ is an unknown function (our aim being to show that $\kappa \equiv 0$). Relation (8) can then be rewritten as

$$\begin{aligned} B^g &= \kappa(W^\pm)_g((J_+J_-)_g) = \kappa(W^\pm)_{g_\pm}((J_+J_-)_{g_\pm}) = \frac{1}{6}\kappa s_\pm(J_+J_-)_{g_\pm} \\ &= \frac{1}{6}\kappa s_\pm^3(J_+J_-)_g. \end{aligned}$$

We deduce that $\kappa s_+^3 = \kappa s_-^3$. Since g_+ and g_- are not homothetic, s_+ and s_- are not identical; since they are Killing potentials, they are then not equal on a dense open set. Thus $\kappa \equiv 0$. \square

4. AMBITORIC GEOMETRY

We are interested in studying ambikähler 4-manifolds where both Kähler metrics are toric with respect to a common T^2 -action. The following definition captures the pointwise geometry of the situation.

Definition 3. An ambikähler 4-manifold (M, c, J_+, J_-) is said to be *ambitoric* if it is equipped with a 2-dimensional family \mathfrak{t} of vector fields which are linearly independent on a dense open set, and are Poisson-commuting hamiltonian Killing vector fields with respect to both Kähler structures $(g_\pm, J_\pm, \omega_\pm)$.

Note that hamiltonian vector fields $K = \text{grad}_\omega f$ and $\tilde{K} = \text{grad}_\omega \tilde{f}$ Poisson commute (i.e., $\{f, \tilde{f}\} = 0$) if and only if they are *lagrangian* in the sense that $\omega(K, \tilde{K}) = 0$; it then follows that K and \tilde{K} commute (i.e., $[K, \tilde{K}] = 0$). Thus \mathfrak{t} is an abelian Lie algebra under Lie bracket of vector fields. We usually refer to potentials for hamiltonian Killing vector fields as *momenta* or *Killing potentials*.

We further motivate the definition by examples in the following subsections.

4.1. Orthotoric Kähler surfaces are ambitoric.

Definition 4. [6] A Kähler surface (M, g, J) is *orthotoric* if it admits two independent hamiltonian Killing vector fields, K_1 and K_2 , with Poisson-commuting momenta $x + y$ and xy , respectively, where x and y are smooth functions on M such that dx and dy are orthogonal.

The following result is an immediate corollary to [6, Props. 8 & 9].

Proposition 3. *Any orthotoric Kähler surface (M, g_+, J_+, K_1, K_2) admits a canonical opposite hermitian structure J_- with respect to which M is ambitoric with $\mathfrak{t} = \langle \{K_1, K_2\} \rangle$.*

4.2. Ambitoric Kähler surfaces of Calabi type.

Definition 5. [6] A Kähler surface (M, g_+, J_+) is said to be of *Calabi type* if it admits a nonvanishing hamiltonian Killing vector field K such that the negative almost-hermitian pair (g_+, J_-) —with J_- equal to J_+ on the distribution spanned by K and J_+K , but $-J_+$ on the orthogonal distribution—is conformally Kähler.

Thus, any Kähler surface of Calabi type is canonically ambikähler. An explicit formula for Kähler metrics of Calabi type, using the LeBrun normal form [36] of a Kähler metric with a hamiltonian Killing vector field, is obtained in [6, Prop. 13]: (g_+, J_+, ω_+) is given locally by

$$(9) \quad \begin{aligned} g_+ &= (az - b)g_\Sigma + w(z)dz^2 + w(z)^{-1}(dt + \alpha)^2, \\ \omega_+ &= (az - b)\omega_\Sigma + dz \wedge (dt + \alpha), \quad d\alpha = a\omega_\Sigma, \end{aligned}$$

where z is the momentum of the Killing vector field, t is a function on M with $dt(K) = 1$, $w(z)$ is function of one variable, g_Σ is a metric on a 2-manifold Σ with area form ω_Σ , α is a 1-form on Σ and a, b are constant.

The second conformal Kähler structure is then given by

$$\begin{aligned} g_- &= (az - b)^{-2}g_+, \\ \omega_- &= (az - b)^{-1}\omega_\Sigma - (az - b)^{-2}dz \wedge (dt + \alpha). \end{aligned}$$

Note that the $(\Sigma, (az - b)\omega_\Sigma, (az - b)g_\Sigma)$ is identified with the *Kähler quotient* of (M, g_+, ω_+) at the value z of the momentum. The following observation is then straightforward to establish.

Proposition 4. *An ambikähler structure of Calabi type is ambitoric with respect to Killing vector fields K_1, K_2 with $K \in \langle \{K_1, K_2\} \rangle$ if and only if $(\Sigma, g_\Sigma, \omega_\Sigma)$ admits a hamiltonian Killing vector field.*

We shall refer to ambitoric 4-manifolds arising locally from Proposition 4 as *ambitoric Kähler surfaces of Calabi type*. A more precise description is as follows.

Definition 6. An ambitoric 4-manifold (M, c, J_+, J_-) is said to be of Calabi type if the corresponding 2-dimensional family of vector fields contains one, say K , with respect to which the Kähler metric (g_+, J_+) (equivalently, (g_-, J_-)) is of Calabi type on the dense open set where K is nonvanishing; without loss, we can then assume that $J_+ = J_-$ on $\langle \{K, J_+K\} \rangle$.

Note that this definition includes the case of a local Kähler product of two Riemann surfaces each admitting a nontrivial Killing vector field (when we have $a = 0$

in (9)). In the non-product case we can assume without loss $a = 1, b = 0$ and the Kähler structure (g_+, ω_+) takes the form

$$(10) \quad \begin{aligned} g_+ &= z g_\Sigma + \frac{z}{V(z)} dz^2 + \frac{V(z)}{z} (dt + \alpha)^2, \\ \omega_+ &= z \omega_\Sigma + dz \wedge (dt + \alpha), \quad d\alpha = \omega_\Sigma, \end{aligned}$$

while the other Kähler metric $(g_- = z^{-2} g_+, J_-)$ is also of Calabi type with respect to $K = \partial/\partial t$, with momentum $\bar{z} = z^{-1}$ and corresponding function $\bar{V}(\bar{z}) = \bar{z}^4 V(1/\bar{z}) = V(z)/z^4$.

The form (10) of a non-product Kähler metric of Calabi type is well adapted to curvature computations. We summarize in the next proposition the local result that we need in this paper.

Proposition 5. *Let (M, g_+, J_+) be a non-product Kähler surface of Calabi type with respect to K . Denote by J_- the corresponding negative hermitian structure and by g_- the conformal Kähler metric with respect to J_- .*

- *(g_+, J_+) is extremal if and only if (g_-, J_-) is extremal and this happens precisely when (Σ, g_Σ) in (10) is of constant Gauss curvature k and $V(z) = a_0 z^4 + a_1 z^3 + k z^2 + a_3 z + a_4$. In particular, (c, J_+, J_-) is locally ambitoric.*
- *The conformal structure is Bach-flat if and only if, in addition, $4a_0 a_4 - a_1 a_3 = 0$.*
- *(g_+, J_+) has constant scalar curvature if and only if it is extremal with $a_0 = 0$; it is Kähler–Einstein if, moreover, $a_3 = 0$.*

Proof. The result is well-known under the extra assumption that the scalar curvature s_+ of the extremal Kähler metric g_+ is a Killing potential for a multiple of K (see e.g., [6, Prop. 14]). However, one can show [9, Prop. 5] that the later assumption is, in fact, necessary for g_+ to be extremal. \square

4.3. Ambihhermitian metrics with diagonal Ricci tensor.

Definition 7. An ambihhermitian metric (g, J_+, J_-) on a 4-manifold M has *diagonal Ricci tensor* if the Ricci tensor $Ric^g \in C^\infty(M, S^2 T^* M)$ is both J_+ and J_- invariant.

It follows that we may write

$$Ric^g(X, Y) = f g(X, Y) + h g(J_+ J_- X, Y)$$

for some functions f, h . Clearly ambihhermitian Einstein metrics have diagonal Ricci tensor. A Riemannian version of the Goldberg–Sachs theory [4] implies that the Weyl tensors W^+ and W^- of *any* ambihhermitian metric with diagonal Ricci tensor are degenerate, and hence the Lee forms θ_\pm^g of J_\pm have the property that $d\theta_+^g$ is antiselfdual, while $d\theta_-^g$ is selfdual.

Let us suppose that $d\theta_\pm^g = 0$, so that (g, J_+, J_-) is locally ambikähler. (This is automatic if M is compact, or if $\theta_+^g + \theta_-^g$ is closed, for example.) Thus on an open set we have Kähler metrics given by $g_\pm = \varphi_\pm^2 g$ with Kähler forms $\omega_\pm = g_\pm(J_\pm \cdot, \cdot)$.

A well-known argument (comparing the J_\pm -invariant Ricci tensors of g and g_\pm) now shows that φ_\pm are Killing potentials with respect to (g_\pm, J_\pm) respectively. The corresponding ω_\pm -hamiltonian Killing vector fields $Z_\pm = \text{grad}_{\omega_\pm} \varphi_\pm$ are also Killing vector fields of g , since they preserve φ_\pm respectively. Hence they also preserve Ric^g , W^+ and W^- .

Let us further suppose that Z_+ preserves J_- , which is automatic unless g is selfdual Einstein, and that Z_- preserves J_+ , which is similarly automatic unless g is antiselfdual Einstein. It follows that Z_\pm preserves φ_\mp up to scale.

Definition 8. A *diagonal ambikähler structure* (g, J_+, J_-) is an ambihermitian metric g with diagonal Ricci tensor which is ambikähler and such that the vector fields Z_\pm defined above are holomorphic and Killing with respect to both (g_+, J_+) and (g_-, J_-) .

It follows that Z_\pm are lagrangian, hence Poisson-commuting hamiltonian Killing vector fields with respect to both (g_+, J_+, ω_+) and (g_-, J_-, ω_-) .

Proposition 6. *Let (g, J, ω) be a diagonal ambikähler structure on connected manifold M with hamiltonian Killing vector fields Z_+, Z_- . Then precisely one of the following cases occurs:*

- (i) Z_+ and Z_- are identically zero and (M, c, J_+, J_-) is a locally a Kähler product of Riemann surfaces;
- (ii) $Z_+ \otimes Z_-$ is identically zero, but Z_+ and Z_- are not, and (M, c, J_+, J_-) is either orthotoric or of Calabi type;
- (iii) $Z_+ \wedge Z_-$ is identically zero, but $Z_+ \otimes Z_-$ is not, and (M, c, J_+, J_-) is either ambitoric or of Calabi type;
- (iv) $Z_+ \wedge Z_-$ is not identically zero, and (M, c, J_+, J_-) is ambitoric.

In particular (M, c, J_+, J_-) is either ambitoric or of Calabi type.

Proof. By connectedness and unique continuation for holomorphic vector fields, these conditions are mutually exclusive and the open condition in each case holds on a dense open set. Case (i) is trivial: here $g = g_+ = g_-$ is Kähler and $J_+ J_-$ is a D^g -parallel product structure.

In case (ii) either Z_+ or Z_- is zero on each component of the dense open set where they are not both zero. Suppose, without loss that $Z_+ = 0$ so that $g = g_+$ and $Z_- = J_- \text{grad}_{g_-} \varphi_- = J_- \text{grad}_g \lambda$ with $\lambda = -1/\varphi_-$. However, since Z_- also preserves ω_+ , $J_+ J_- d\lambda$ is closed, hence locally equal to $\frac{1}{2} d\sigma$ for a smooth function σ . According to [6, Remark 2], the 2-form $\varphi := \frac{3}{2} \sigma \omega_+ + \lambda^3 \omega_-$ is hamiltonian with respect to the Kähler metric (g_+, J_+) ; by [6, Theorems 1 & 3], this means that $g = g_+$ is either orthotoric (on an open dense subset of M), or is of Calabi type.

In case (iii) Z_+ and Z_- are linearly dependent, but are both nonvanishing on a dense open set. Hence, we may assume, up to rescaling on each component of this dense open set, that $Z := Z_+ = Z_-$. This is equivalent to

$$(11) \quad J_+ \left(\frac{d\varphi_+}{\varphi_+^2} \right) = J_- \left(\frac{d\varphi_-}{\varphi_-^2} \right),$$

which may be rewritten as

$$2J_\pm d \left(\frac{1}{\varphi_+ \varphi_-} \right) = J_\mp d \left(\frac{1}{\varphi_+^2} + \frac{1}{\varphi_-^2} \right).$$

Since hg , with $h = 1/\varphi_+ \varphi_-$, is the barycentre of g_+ and g_- , it follows (cf. [34] and Appendix A.2) that the symmetric tensor $g(S \cdot, \cdot)$, where $S = fId + hJ_+ J_-$ and $2f = 1/\varphi_+^2 + 1/\varphi_-^2$, is a Killing tensor with respect to g . Clearly $\mathcal{L}_Z S = 0$, and it follows from (11) that $D^g Z^b$ is both J_+ and J_- invariant. Thus $X \mapsto D_X^g Z$ commutes with S and $D_Z^g S = 0$. Straightforward computations now show that SZ is a Killing field with respect to g , and hamiltonian with respect to ω_\pm .

Moreover, Z and SZ commute and span a lagrangian subspace with respect to ω_\pm , so define an ambitoric structure on the open set where they are linearly independent. Clearly Z and SZ are linearly dependent only where $J_+ J_- Z$ is proportional to Z , in which case g_\pm is of Calabi type.

Case (iv) follows by definition. \square

4.4. Ambihhermitian Einstein 4-manifolds are locally ambitoric. According to Proposition 1, any Einstein metric g with degenerate half Weyl tensors W^\pm —in particular, any ambihhermitian Einstein metric (g, J_+, J_-) —is ambikähler and Bach-flat. Conversely by Proposition 2, Bach-flat ambikähler metrics (g_\pm, J_\pm) are conformal to an Einstein metric g on a dense open set.

In the generic case that W^\pm are both nonzero, the ambikähler metrics conformal to g are $g_\pm = |W^\pm|_g^{2/3} g$, and the Einstein metric is recovered up to homothety as $g = s_\pm^{-2} g_\pm$, where s_\pm is the scalar curvature of g_\pm . We have already noted that the vector fields $Z_\pm := J_\pm \text{grad}_{g_\pm} s_\pm$ are Killing with respect to g_\pm (respectively) and hence also g . More is true.

Proposition 7. *Let (M, c, J_+, J_-) be a Bach-flat ambikähler manifold such that the Kähler metrics g_\pm have nonvanishing scalar curvatures s_\pm . Then the vector fields $Z_\pm = J_\pm \text{grad}_{g_\pm} s_\pm$ are each Killing with respect to both g_+ and g_- , holomorphic with respect to both J_+ and J_- , and lagrangian with respect to both ω_+ and ω_- (i.e., $\omega_\pm(Z_+, Z_-) = 0$); in particular Z_+ and Z_- commute.*

Furthermore (M, c, J_+, J_-) is ambitoric in a neighbourhood of any point in a dense open subset, and on a neighbourhood of any point where Z_+ and Z_- are linearly independent, we may take $\mathfrak{t} = \langle \{Z_+, Z_-\} \rangle$.

Proof. Z_+ and Z_- are conformal vector fields, so they preserve W^\pm and its unique simple eigenspaces. One readily concludes [4, 22] that the Lie derivatives of g_+ , g_- , J_+ , J_- (and hence also ω_+ and ω_-) all vanish. Consequently, $\mathcal{L}_{Z_+} s_- = 0 = \mathcal{L}_{Z_-} s_+$ —or equivalently $\omega_\pm(Z_+, Z_-) = 0$. This proves the first part.

Since we are now in the situation of Proposition 6, it remains to show that (M, c, J_+, J_-) is ambitoric even in cases where Proposition 6 only asserts that the structure has Calabi type. In case (i) this is easy: $g = g_+ = g_-$ is Kähler–Einstein with D^g -parallel product structure, so is the local product of two Riemann surfaces with constant Gauss curvatures.

In case (ii) $g = g_+$ is Kähler–Einstein, Proposition 5 implies that the quotient Riemann surface (Σ, g_Σ) has constant Gauss curvature.

In case (iii) g_\pm are extremal, so we have either a local product of two extremal Riemann surfaces or, by Proposition 5, the corresponding quotient Riemann surface (Σ, g_Σ) has constant Gauss curvature; it follows that g_+ is locally ambitoric of Calabi type. \square

Remark 2. The case $Z_+ = 0$ above yields the following observation of independent interest: let (M, g, J, ω) be a Kähler–Einstein 4-manifold with everywhere degenerate antiselfdual Weyl tensor W^- , and trivial first deRham cohomology group. Then (M, g, J, ω) admits a globally defined hamiltonian 2-form in the sense of [6] and, on an open dense subset M^0 , the metric is one of the following: a Kähler product metric of two Riemann surfaces of equal constant Gauss curvatures, or a Kähler–Einstein metric of Calabi type, described in Proposition 5, or a Kähler–Einstein ambitoric metric of parabolic type (see section 6.3).

Theorem 1. *Let (M, g) be a connected oriented riemannian Einstein 4-manifold with degenerate half-Weyl tensors W^+ and W^- . Then, about each point of an open dense subset of M , g admits a compatible ambitoric structure (J_+, J_-, \mathfrak{t}) .*

Proof. If W^+ and W^- identically vanish, we have a real space form and g is locally conformally-flat (and is obviously locally ambitoric).

If g is half-conformally-flat but not flat, then g admits a canonically defined hermitian structure $J = J_+$, i.e. g is an Einstein, hermitian self-dual metric (see [5])

for a classification). In particular, g is an Einstein metrics in the conformal classes of a self-dual (or, equivalently, Bochner-flat) Kähler metric (g_+, J_+) . We learn from [12, 6] that such a Kähler metric must be either orthotoric or of Calabi type over a Riemann surface (Σ, g_Σ) of constant Gauss curvature. In both cases the metric is locally ambitoric by the examples discussed in the previous subsections.

In the generic case, the result follows from Propositions 1, 2 and 7. \square

5. LOCAL CLASSIFICATION OF AMBITORIC STRUCTURES

In order to classify ambitoric structures on the dense open set where the (local) torus action is free, we let (M, c, J_+, J_-) denote a connected, simply-connected, ambihermitian 4-manifold and $\mathbf{K}: \mathfrak{t} \rightarrow C^\infty(M, TM)$ a 2-dimensional family of pointwise linearly independent vector fields. Let $\varepsilon \in \wedge^2 \mathfrak{t}^*$ be a fixed area form.

5.1. Holomorphic lagrangian torus actions. We denote by K_λ the image of $\lambda \in \mathfrak{t}$ under \mathbf{K} , by \mathfrak{t}_M the rank 2 subbundle of TM spanned by these vector fields, and by $\theta \in \Omega^1(M, \mathfrak{t})$ the \mathfrak{t} -valued 1-form vanishing on $\mathfrak{t}_M^\perp \subset TM$ with $\theta(K_\lambda) = \lambda$.

We first impose the condition that \mathbf{K} is an infinitesimal J_\pm -holomorphic and ω_\pm -lagrangian torus action. We temporarily omit the \pm subscript, since we are studying the complex structures separately. The lagrangian condition means that \mathfrak{t}_M is orthogonal and complementary to its image $J\mathfrak{t}_M$ under J ; thus $J\mathfrak{t}_M = \mathfrak{t}_M^\perp$. The remaining conditions (including the integrability of J) imply that the vector fields $\{K_\lambda : \lambda \in \mathfrak{t}\}$ and $\{JK_\lambda : \lambda \in \mathfrak{t}\}$ all commute under Lie bracket, or equivalently that the dual 1-forms θ and $J\theta$ are both closed. Thus we may write $\theta = d\mathbf{t}$ with $dd^c \mathbf{t} = 0$, where $d^c \mathbf{t} = Jd\mathbf{t}$ and the “angular coordinate” $\mathbf{t}: M \rightarrow \mathfrak{t}$ is defined up to an additive constant. Conversely, if $dd^c \mathbf{t} = 0$ then $d\mathbf{t} - \sqrt{-1}d^c \mathbf{t}$ generates a closed differential ideal $\Omega^{(1,0)}$ for J so that J is integrable.

5.2. Regular ambitoric structures. We now combine this analysis for the complex structures J_\pm . It follows that $J_+ \mathfrak{t}_M$ and $J_- \mathfrak{t}_M$ coincide and that \mathfrak{t}_M is preserved by the involution $-J_+ J_-$. Since the eigenbundles (pointwise eigenspaces) of $-J_+ J_-$ are J_\pm -invariant, \mathfrak{t}_M cannot be an eigenbundle and hence decomposes into $+1$ and -1 eigenbundles ξ_M and η_M : the line bundles ξ_M , η_M , $J_+ \xi_M = J_- \xi_M$ and $J_+ \eta_M = J_- \eta_M$ provide an orthogonal direct sum decomposition of TM .

We denote the images of ξ_M and η_M under $d\mathbf{t}$ by ξ and η respectively. We thus obtain a smooth map $(\xi, \eta): M \rightarrow \mathbb{P}(\mathfrak{t}) \times \mathbb{P}(\mathfrak{t}) \setminus \Delta(\mathfrak{t})$ where $\Delta(\mathfrak{t})$ is the diagonal.

The derivatives $d\xi \in \Omega^1(M, \xi^* TP(\mathfrak{t}))$ and $d\eta \in \Omega^1(M, \eta^* TP(\mathfrak{t}))$ vanish on \mathfrak{t}_M (since ξ and η are \mathfrak{t} -invariant). In fact, more is true: they span orthogonal directions in T^*M . (Note that $\xi^* TP(\mathfrak{t}) \cong \text{Hom}(\xi, \underline{\mathfrak{t}}/\xi)$, with $\underline{\mathfrak{t}} := M \times \mathfrak{t}$, is a line bundle on M , and similarly for $\eta^* TP(\mathfrak{t})$.)

Lemma 3. *$d\xi$ vanishes on $J_\pm \eta_M$ and $d\eta$ vanishes on $J_\pm \xi_M$; hence $0 = d\xi \wedge d\eta \in \Omega^2(M, \xi^* TP(\mathfrak{t}) \otimes \eta^* TP(\mathfrak{t}))$ only on the subset of M where $d\xi = 0$ or $d\eta = 0$.*

Proof. The 1-form $(J_+ + J_-)d\mathbf{t}$ is closed, vanishes on $J_\pm \eta_M$ and \mathfrak{t}_M , and takes values in $\xi \subset \underline{\mathfrak{t}}$ (it is nonzero on $J_\pm \xi_M$). Hence for any section u of ξ ,

$$0 = d(\varepsilon(u, (J_+ + J_-)d\mathbf{t})) = \varepsilon(du \wedge (J_+ + J_-)d\mathbf{t})$$

and so $(du \bmod \xi) \wedge (J_+ + J_-)d\mathbf{t} = 0$. This implies that $d\xi$ is a multiple F of $\frac{1}{2}(J_+ + J_-)d\mathbf{t} \in \Omega^1(M, \xi)$. Similarly $d\eta$ is a multiple G of $\frac{1}{2}(J_+ - J_-)d\mathbf{t}$. \square

Thus if ξ and η are functionally dependent on an connected open set U , then one of them is constant (spanned by $\lambda \in \mathfrak{t}$); if M is ambitoric, it is then of Calabi type (with respect to K_λ). In the ambitoric case, there is thus a dense open set M^0

such that on each connected component, the ambitoric structure is either of Calabi type, or $d\xi \wedge d\eta$ is nonvanishing.

Definition 9. If $d\xi$ and $d\eta$ nowhere vanish, we say $(M, c, J_+, J_-, \mathbf{K})$ is *regular*.

In the regular case $d\xi = \frac{1}{2}F(\xi)(J_+ + J_-)dt$ and $d\eta = \frac{1}{2}G(\eta)(J_+ - J_-)dt$ where F, G are local sections of $\mathcal{O}(3)$ over $P(\mathfrak{t})$; more precisely $F(\xi): M \rightarrow \text{Hom}(\xi, \xi^*TP(\mathfrak{t}))$ and similarly for $G(\eta)$, but $TP(\mathfrak{t}) \cong \mathcal{O}(2)$ using ε . We let ξ^\natural denote the composite of ξ with the natural section of $\mathcal{O}(1) \otimes \mathfrak{t}$ over $P(\mathfrak{t})$, and similarly η^\natural . We construct from these J_\pm -related orthogonal 1-forms

$$\frac{d\xi}{F(\xi)}, \quad \frac{\varepsilon(dt, \eta^\natural)}{\varepsilon(\xi^\natural, \eta^\natural)}, \quad \frac{d\eta}{G(\eta)}, \quad \frac{\varepsilon(\xi^\natural, dt)}{\varepsilon(\xi^\natural, \eta^\natural)}$$

(with values in the line bundles ξ^* or η^*) which may be used to write any \mathfrak{t} -invariant metric g in the conformal class as

$$\frac{d\xi^2}{F(\xi)U(\xi, \eta)} + \frac{d\eta^2}{G(\eta)V(\xi, \eta)} + \frac{F(\xi)}{U(\xi, \eta)} \left(\frac{\varepsilon(dt, \eta^\natural)}{\varepsilon(\xi^\natural, \eta^\natural)} \right)^2 + \frac{G(\eta)}{V(\xi, \eta)} \left(\frac{\varepsilon(\xi^\natural, dt)}{\varepsilon(\xi^\natural, \eta^\natural)} \right)^2.$$

Here U and V are local sections of $\mathcal{O}(1, 0)$ and $\mathcal{O}(0, 1)$ over $P(\mathfrak{t}) \times P(\mathfrak{t}) \setminus \Delta(\mathfrak{t})$.

More concretely, in a neighbourhood of any point, a basis $\lambda = 1, 2$ for \mathfrak{t} may be chosen to provide an affine chart for $P(\mathfrak{t})$ so that $K_\xi := \xi K_1 - K_2$ and $K_\eta := \eta K_1 - K_2$ are sections of ξ_M and η_M respectively, where $\xi > \eta$ are functionally independent coordinates on M . The components of $\mathbf{t}: M \rightarrow \mathfrak{t}$ in this basis complete a coordinate system (ξ, η, t_1, t_2) with coordinate vector fields

$$\frac{\partial}{\partial \xi} = \frac{J_+ K_\xi}{F(\xi)}, \quad \frac{\partial}{\partial \eta} = \frac{J_+ K_\eta}{G(\eta)}, \quad \frac{\partial}{\partial t_1} = K_1, \quad \frac{\partial}{\partial t_2} = K_2.$$

Replacing (J_+, J_-) with $(-J_+, -J_-)$ if necessary, we can assume without loss that F and G (now functions of one variable) are both positive, and thus obtain the following description of \mathfrak{t} -invariant ambihermitian metrics in the conformal class.

Proposition 8. *An ambihermitian metric (g, J_+, J_-) which is regular with respect to a 2-dimensional family of commuting, J_\pm -holomorphic lagrangian Killing vector fields is given locally by*

$$(12) \quad g = \frac{d\xi^2}{F(\xi)U(\xi, \eta)} + \frac{d\eta^2}{G(\eta)V(\xi, \eta)} + \frac{F(\xi)(dt_1 + \eta dt_2)^2}{U(\xi, \eta)(\xi - \eta)^2} + \frac{G(\eta)(dt_1 + \xi dt_2)^2}{V(\xi, \eta)(\xi - \eta)^2},$$

$$(13) \quad \omega_\pm^g = \frac{d\xi \wedge (dt_1 + \eta dt_2)}{U(\xi, \eta)(\xi - \eta)} \pm \frac{d\eta \wedge (dt_1 + \xi dt_2)}{V(\xi, \eta)(\xi - \eta)},$$

$$(14) \quad d_+^c \xi = d_-^c \xi = F(\xi) \frac{dt_1 + \eta dt_2}{\xi - \eta}, \quad d_+^c \eta = -d_-^c \eta = G(\eta) \frac{dt_1 + \xi dt_2}{\xi - \eta}$$

for some positive functions U and V of two variables, and some positive functions F and G of one variable. (Here and later, $d_\pm^c h = J_\pm dh$ for any function h .)

We now impose the condition that (c, J_+) and (c, J_-) admit \mathfrak{t} -invariant Kähler metrics g_+ and g_- . Let f be the conformal factor relating g_\pm by $g_- = f^2 g_+$. Clearly f is \mathfrak{t} -invariant and so, therefore, is the metric

$$g_0 := f g_+ = f^{-1} g_-$$

which we call the *barycentric metric* of the ambitoric structure. The Lee forms, θ_\pm^0 , of (g_0, J_\pm) are given by $\theta_\pm^0 = \mp \frac{1}{2} \log f$. Conversely, suppose there is an invariant

ambihhermitian metric g_0 in the conformal class whose Lee forms θ_\pm^0 satisfy

$$(15) \quad \theta_+^0 + \theta_-^0 = 0$$

$$(16) \quad d(\theta_+^0 - \theta_-^0) = 0.$$

Then writing locally $\theta_+^0 = -\frac{1}{2}d \log f = -\theta_-^0$ for some positive function f , the metrics $g_\pm := f^{\mp 1}g_0$ are Kähler with respect to J_\pm respectively.

Thus, regular ambitoric conformal structures are defined by ambihhermitian metrics g_0 given locally by Lemma 8, and whose Lee forms θ_\pm^0 satisfy (15) and (16).

Lemma 4. *For an ambihhermitian metric given by Lemma 8 the relation (15) is satisfied (with $g_0 = g$) if and only if $U = U(\xi)$ is independent of η and $V = V(\eta)$ is independent of ξ . In this case (16) is equivalent to $U(\xi)^2 = R(\xi)$ and $V(\eta)^2 = R(\eta)$, where $R(s) = r_0 s^2 + 2r_1 s + r_2$ is a polynomial of degree at most two.*

Under both conditions, the conformal factor f with $g_- = f^2 g_+$ is given—up to a constant multiple—by

$$(17) \quad f(\xi, \eta) = \frac{R(\xi)^{1/2} R(\eta)^{1/2} + R(\xi, \eta)}{\xi - \eta}$$

where

$$\begin{aligned} R(\xi, \eta) &= R(\xi) - \frac{1}{2}(\xi - \eta)R'(\xi) = R(\eta) + \frac{1}{2}(\xi - \eta)R'(\eta) \\ &= r_0 \xi \eta + r_1(\xi + \eta) + r_2, \end{aligned}$$

Proof. The Lee forms θ_\pm^g are given by $2\theta_\pm^g = u_\pm d\xi + v_\pm d\eta$, with

$$u_\pm = \frac{V_\xi}{V} \pm \frac{V}{(\xi - \eta)U}, \quad v_\pm = \frac{U_\eta}{U} \mp \frac{U}{(\xi - \eta)V}.$$

In particular, $u_+ + u_- = 2V_\xi/V$ and $v_+ + v_- = 2U_\eta/U$. It follows that $\theta_+^g + \theta_-^g = 0$ if and only if $U_\eta = 0$ and $V_\xi = 0$. This proves the first part of the lemma.

If (15) is satisfied, then

$$\theta_+^g = \frac{1}{2} \left(\frac{V(\eta)}{(\xi - \eta)U(\xi)} d\xi - \frac{U(\xi)}{(\xi - \eta)V(\eta)} d\eta \right).$$

It follows that $d\theta_+^g = 0$ if and only if

$$(18) \quad 2U^2(\xi) - (\xi - \eta)(U^2)'(\xi) = 2V^2(\eta) + (\xi - \eta)(V^2)'(\eta)$$

where $U^2(\xi) = U(\xi)^2$ and $V^2(\eta) = V(\eta)^2$. Differentiating twice with respect to ξ , we obtain $(\xi - \eta)(U^2)'''(\xi) = 0$, and similarly $(\xi - \eta)(V^2)'''(\eta) = 0$. Thus U^2 and V^2 are both polynomials of degree at most two. We may now set $\xi = \eta$ in (18) to conclude that U^2 and V^2 coincide. Without loss of generality, we assume that U and V are both positive everywhere, so that $U(\xi) = R(\xi)^{1/2}$ and $V(\eta) = R(\eta)^{1/2}$ for a polynomial R of degree at most two. By using the identity

$$R(\xi) - R(\eta) - \frac{1}{2}(\xi - \eta)(R'(\xi) + R'(\eta)) \equiv 0$$

we easily check (17). \square

Note that the quadratic R is, more invariantly, a homogeneous polynomial of degree 2 on \mathfrak{t} (an algebraic section of $\mathcal{O}(2)$). However the parameterization of ambitoric structures by R and the local sections F and G of $\mathcal{O}(3)$ is not effective because of the $\mathrm{SL}(\mathfrak{t})$ symmetry and homothety freedom in the metric. Modulo this freedom, there are only three distinct cases for R : no real roots ($r_1^2 < r_0 r_2$), one real root ($r_1^2 = r_0 r_2$) and two real roots ($r_1^2 > r_0 r_2$). We shall later refer to these cases as *elliptic*, *parabolic* and *hyperbolic* respectively.

Remark 3. The emergence of a homogeneous polynomial of degree 2 on \mathfrak{t} merits a more conceptual explanation. It also seems to be connected with a curious symmetry breaking phenomenon between ω_+ and ω_- . In (13), ω_{\pm}^g are interchanged on replacing V by $-V$. This is compatible with the equality $U^2 = V^2$ derived in the above lemma. However, the choice of square root of R to satisfy the positivity condition breaks this symmetry.

5.3. Local classification in adapted coordinates. The square root in the general form of an ambitoric metric is somewhat awkward: although we are interested in real riemannian geometry, the complex analytic continuation of the metric will be branched. This suggests pulling back the metric to a branched cover and making a coordinate change to eliminate the square root. This is done by introducing rational functions ρ and σ of degree 2 such that

$$(19) \quad R(\sigma(z)) = \rho(z)^2.$$

If we then write $\xi = \sigma(x)$, $\eta = \sigma(y)$, $A(x) = F(\sigma(x))\rho(x)/\sigma'(x)^2$ and $B(y) = G(\sigma(y))\rho(y)/\sigma'(y)^2$, the barycentric metric may be rewritten as

$$(20) \quad g_0 = \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x) \left(\frac{\sigma'(x)(dt_1 + \sigma(y)dt_2)}{(\sigma(x) - \sigma(y))\rho(x)} \right)^2 + B(y) \left(\frac{\sigma'(y)(dt_1 + \sigma(x)dt_2)}{(\sigma(x) - \sigma(y))\rho(y)} \right)^2.$$

There are many solutions to (19). We seek a family that covers all three cases for R and yields metrics that are amenable to computation. We do this by solving the equation geometrically. Let W be a 2-dimensional real vector space equipped (for convenience) with a symplectic form κ (a non-zero element of $\wedge^2 W^*$). This defines an isomorphism $W \rightarrow W^*$ sending $u \in W$ to the linear form $u^\flat: v \mapsto \kappa(u, v)$; similarly there is a Lie algebra isomorphism from $\mathfrak{sl}(W)$ (the trace-free endomorphisms of W) to $S^2 W^*$ (the quadratic forms on W , under a normalized Poisson bracket $\{, \}$) sending $a \in \mathfrak{sl}(W)$ to the quadratic form $u \mapsto \kappa(a(u), u)$.

The quadratic form $-\det$ on $\mathfrak{sl}(W)$ induces a quadratic form Q on $S^2 W^*$ proportional to the discriminant, which polarizes to give an $\mathfrak{sl}(W)$ -invariant inner product \langle, \rangle of signature $(2, 1)$. The following identity is readily checked:

$$(21) \quad Q(\{p, \tilde{p}\}) = 4(\langle p, \tilde{p} \rangle^2 - Q(p)Q(\tilde{p})).$$

Our construction proceeds by introducing a quadratic form q on W , i.e., an element of $S^2 W^*$. The Poisson bracket $\{q, \cdot\}: S^2 W^* \rightarrow S^2 W^*$ vanishes on the span of q and its image is the 2-dimensional subspace $S_{0,q}^2 W^*$ orthogonal to q . We thus obtain a map

$$\text{ad}_q: S^2 W^* / \langle q \rangle \rightarrow S_{0,q}^2 W^*.$$

We now define $\sigma_q: W^* \rightarrow S^2 W^* / \langle q \rangle$ via the Veronese map

$$(22) \quad \sigma_q(\alpha) = \alpha \otimes \alpha \pmod{q}$$

and let $R_q = \text{ad}_q^* Q$. Thus $R_q(\sigma_q(\alpha)) = Q(\{q, \alpha \otimes \alpha\}) = 4\langle q, \alpha \otimes \alpha \rangle^2$ by (21) (with $p = q$ and $\tilde{p} = \alpha \otimes \alpha$, which is null) and so

$$(23) \quad R_q(\sigma_q(\alpha)) = \rho_q(\alpha)^2,$$

where $\rho_q(\alpha) = q(\alpha^\sharp)$ and $\alpha^\sharp \in W$ is determined by $(\alpha^\sharp)^\flat = \alpha$.

A geometrical solution to (19) is now given by identifying \mathfrak{t} with $S^2 W^* / \langle q \rangle$, and R with R_q . Note that R_q is positive definite if $Q(q) < 0$, signature $(1, 1)$ if $Q(q) > 0$, or semi-positive degenerate if $Q(q) = 0$. Thus R_q can have arbitrary type

(elliptic, parabolic or hyperbolic). This geometrical solution represents ξ as $\sigma_q(\mathbf{x})$ and η as $\sigma_q(\mathbf{y})$, where

$$(\mathbf{x}, \mathbf{y}): M \rightarrow \mathbf{P}(W^*) \times \mathbf{P}(W^*) \setminus \Delta(W^*).$$

For $Q(q) \neq 0$, σ_q defines a branched double cover of $\mathbf{P}(\mathfrak{t})$ by $\mathbf{P}(W^*)$. For $Q(q) = 0$, the projective transformation appears to be singular for $q \in \langle \alpha \otimes \alpha \rangle$, but this singularity is removable (by sending such α to $\langle \alpha \rangle \odot W^* \bmod q$) and σ_q identifies $\mathbf{P}(W^*)$ with $\mathbf{P}(\mathfrak{t})$ via the pencil of lines through a point on a conic.

An area form $\varepsilon \in \wedge^2 \mathfrak{t}^*$ is given by $\varepsilon(\lambda, \mu) = \langle \text{ad}_q \lambda, \mu \rangle$. In particular

$$\varepsilon(\sigma_q(\alpha), \sigma_q(\beta)) = \langle \{q, \alpha \otimes \alpha\}, \beta \otimes \beta \rangle = \kappa(\alpha^\sharp, \beta^\sharp) q(\alpha^\sharp, \beta^\sharp).$$

It follows that the barycentric metric g_0 may be written invariantly as

$$\frac{d\mathbf{x}^2}{A(\mathbf{x})} + \frac{d\mathbf{y}^2}{B(\mathbf{y})} + A(\mathbf{x}) \left(\frac{\langle d\boldsymbol{\tau}, \mathbf{y}^\sharp \otimes \mathbf{y}^\sharp \rangle}{\kappa(\mathbf{x}^\sharp, \mathbf{y}^\sharp) q(\mathbf{x}^\sharp, \mathbf{y}^\sharp)} \right)^2 + B(\mathbf{y}) \left(\frac{\langle d\boldsymbol{\tau}, \mathbf{x}^\sharp \otimes \mathbf{x}^\sharp \rangle}{\kappa(\mathbf{x}^\sharp, \mathbf{y}^\sharp) q(\mathbf{x}^\sharp, \mathbf{y}^\sharp)} \right)^2,$$

where A, B are local sections of $\mathcal{O}(4)$ over $\mathbf{P}(W^*)$, $d\boldsymbol{\tau} = \{q, d\mathbf{t}\}$ and $()^\sharp$ denotes the natural section of $\mathcal{O}(1) \otimes W^* \cong \mathcal{O}(1) \otimes W$ (using κ). Note that $\langle q, d\boldsymbol{\tau} \rangle = 0$.

A more concrete expression may be obtained by introducing a symplectic basis e_1, e_2 of W (so that $\kappa(e_1, e_2) = 1$) and hence an affine coordinate z on $\mathbf{P}(W)$. A quadratic form $q \in S^2 W^*$ may then be written

$$q(z) = q_0 z^2 + 2q_1 z + q_2$$

with polarization

$$q(x, y) = q_0 xy + q_1(x + y) + q_2.$$

In these coordinates the Poisson bracket of $q(z)$ with $w(z)$ is

$$(24) \quad \{q, w\}(z) = q'(z)w(z) - w'(z)q(z) \quad \text{with}$$

$$(25) \quad \{q, w\}_0 = 2q_0 p_1 - 2q_1 p_0, \quad \{q, w\}_1 = q_0 p_2 - q_2 p_0, \quad \{q, w\}_2 = 2q_1 p_2 - 2q_2 p_1,$$

and the quadratic form and inner product on $S^2 W^*$ are

$$Q(q) = q_1^2 - q_0 q_2 \quad \text{and} \quad \langle q, p \rangle = q_1 p_1 - \frac{1}{2}(q_2 p_0 + q_0 p_2).$$

Elements of \mathfrak{t} may be represented by triples $[w] = [w_0, w_1, w_2]$ modulo q , or by the corresponding elements $p = (p_0, p_1, p_2)$ of $S_{0,q}^2 W^*$ where $p = \{q, w\}$. The corresponding vector field on M will be denoted $K^{[w]}$ or $K^{(p)}$, so that $d\mathbf{t}(K^{[w]}) = [w]$ and $d\boldsymbol{\tau}(K^{(p)}) = p$.

Theorem 2. *Let $(M, c, J_+, J_-, \mathfrak{t})$ be an ambitoric 4-manifold with barycentric metric g_0 and Kähler metrics (g_+, ω_+) and (g_-, ω_-) . Then, about any point in an open dense subset of M , there is a neighbourhood in which (c, J_+, J_-) is either of Calabi type with respect to some $\lambda \in \mathfrak{t}$, or there there are \mathfrak{t} -invariant functions x, y , a quadratic polynomial $q(z) = q_0 z^2 + 2q_1 z + q_2$, and functions $A(z)$ and $B(z)$ of one variable with respect to which:*

$$(26) \quad g_0 = \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + A(x) \left(\frac{y^2 d\tau_0 + 2y d\tau_1 + d\tau_2}{2(x-y)q(x,y)} \right)^2 + B(y) \left(\frac{x^2 d\tau_0 + 2x d\tau_1 + d\tau_2}{2(x-y)q(x,y)} \right)^2,$$

$$\begin{aligned}
 (27) \quad \omega_+ &= \frac{2(x-y)}{q(x,y)} \left(\frac{dx \wedge d_+^c x}{A(x)} + \frac{dy \wedge d_+^c y}{B(y)} \right) \\
 &= \frac{dx \wedge (y^2 d\tau_0 + 2y d\tau_1 + d\tau_2) + dy \wedge (x^2 d\tau_0 + 2x d\tau_1 + d\tau_2)}{q(x,y)^2},
 \end{aligned}$$

$$\begin{aligned}
 (28) \quad \omega_- &= \frac{2q(x,y)}{x-y} \left(\frac{dx \wedge d_-^c x}{A(x)} + \frac{dy \wedge d_-^c y}{B(y)} \right) \\
 &= \frac{dx \wedge (y^2 d\tau_0 + 2y d\tau_1 + d\tau_2) - dy \wedge (x^2 d\tau_0 + 2x d\tau_1 + d\tau_2)}{(x-y)^2}.
 \end{aligned}$$

where $2q_1 d\tau_1 = q_0 d\tau_2 + q_2 d\tau_0$ and $q(x,y) = q_0 xy + q_1(x+y) + q_2$.

Conversely, for any data as above, the above metric and Kähler forms does define an ambitoric Kähler structure on any simply connected open set where ω_{\pm} are nondegenerate and g_0 is positive definite.

Proof. The fact that regular ambitoric conformal structures have this form follows easily from Lemmas 8 and 4. One can either substitute into the invariant form of the metric, or carry out the coordinate transformation explicitly using (20). We deduce from (14) that

$$\begin{aligned}
 (29) \quad d_+^c x &= d_-^c x = \frac{A(x)}{2(x-y)q(x,y)}(y^2 d\tau_0 + 2y d\tau_1 + d\tau_2), \\
 d_+^c y &= -d_-^c y = \frac{B(y)}{2(x-y)q(x,y)}(x^2 d\tau_0 + 2x d\tau_1 + d\tau_2).
 \end{aligned}$$

The computation of the conformal factor

$$(30) \quad f(x,y) = \frac{q(x,y)}{x-y}$$

with $\omega_- = f^2 \omega_+$ requires a little more work, but it is straightforward to check that ω_{\pm} are closed, and hence also deduce conversely that any metric of this form is ambitoric. \square

The computation of the momenta, μ^{\pm} , of the \mathfrak{t} -action relatively to ω_{\pm} is quite simple in the case of ω_- , because it has an explicit local primitive:

$$\omega_- = -d\chi, \quad \chi = \frac{xy d\tau_0 + (x+y)d\tau_1 + d\tau_2}{x-y}.$$

Hence a Killing potential for $K^{(p)}$ is

$$(31) \quad \mu_-^{(p)} = \frac{p_0 xy + p_1(x+y) + p_2}{x-y},$$

where $2p_1 q_1 = p_0 q_2 + p_2 q_0$, but we may add an arbitrary constant of integration.

No such explicit potential exists for ω_+ since its closedness depends on the implicit relation $\langle q, d\tau \rangle = 0$. However, since $d\tau = \{q, d\mathbf{t}\}$, it follows that

$$-\iota_{K[w]}\omega_+ = -\frac{\{q, w\}(y) dx + \{q, w\}(x) dy}{2q(x,y)}$$

which is equal to $d\mu_+^{[w]}$ where

$$(32) \quad \mu_+^{[w]} = \frac{w_0 xy + w_1(x+y) + w_2}{q(x,y)}.$$

In this case the constant of integration in this Killing potential for $K^{[w]}$ is expressed by the freedom to add a multiple of (q_0, q_1, q_2) to (w_0, w_1, w_2) .

6. EXTREMAL AND CONFORMALLY EINSTEIN AMBITORIC SURFACES

We now compute the Ricci forms and scalar curvatures of a regular ambitoric Kähler surface, and hence give a local classification of extremal ambitoric structures. By considering the Bach tensor, we also identify the regular ambitoric structures which are conformally Einstein.

6.1. Ricci forms and scalar curvatures. As in [6], we adopt a standard method for computing the Ricci form of a Kähler metric as the curvature of the connection on the canonical bundle: the log ratio of the symplectic volume to any holomorphic volume is a Ricci potential. For regular ambitoric metrics, $d\tau_j + \sqrt{-1}d_\pm^c \tau_j$ ($j = 0, 1, 2$) are J_\pm -holomorphic 1-forms. Using (29) and $\langle q, d\tau \rangle = 0$, we find

$$\begin{aligned} d_\pm^c \tau_0 &= \frac{q_0 x + q_1}{A(x)} dx \mp \frac{q_0 y + q_1}{B(y)} dy, \\ d_\pm^c \tau_1 &= -\frac{q_0 x^2 - q_2}{2A(x)} dx \pm \frac{q_0 y^2 - q_2}{2B(y)} dy, \\ d_\pm^c \tau_2 &= -\frac{q_1 x^2 + q_2 x}{A(x)} dx \pm \frac{q_1 y^2 + q_2 y}{B(y)} dy. \end{aligned}$$

Choosing two linearly independent holomorphic 1-forms among these, we find that

$$v_0 = \frac{(x-y)^2 q(x, y)^2}{A(x)^2 B(y)^2} dx \wedge d_+^c x \wedge dy \wedge d_+^c y.$$

can be taken as a holomorphic volume for both J_+ and J_- (up to sign). The symplectic volumes v_\pm of ω_\pm are

$$\begin{aligned} v_+ &= \frac{(x-y)^2}{q(x, y)^2 A(x) B(y)} dx \wedge d_+^c x \wedge dy \wedge d_+^c y, \\ v_- &= \frac{q(x, y)^2}{(x-y)^2 A(x) B(y)} dx \wedge d_-^c x \wedge dy \wedge d_-^c y. \end{aligned}$$

Hence the Ricci forms $\rho_\pm = -\frac{1}{2} dd_\pm^c \log |v_\pm / v_0|$ of ω_\pm are given by

$$\rho_+ = -\frac{1}{2} dd_+^c \log \frac{A(x) B(y)}{q(x, y)^4}, \quad \rho_- = -\frac{1}{2} dd_-^c \log \frac{A(x) B(y)}{(x-y)^4}.$$

The 2-forms $dd^c x$ and $dd^c y$ are obtained by differentiating the two sides of (29). After some work, we obtain

$$\begin{aligned} dd_\pm^c x &= \left(A'(x) - \frac{q(x) - q_0 (x-y)^2}{(x-y)q(x, y)} A(x) \right) \frac{dx \wedge d_\pm^c x}{A(x)} \\ &\quad \pm \frac{q(y) A(x)}{(x-y)q(x, y)} \frac{dy \wedge d_\pm^c y}{B(y)}, \\ dd_\pm^c y &= \mp \frac{q(x) B(y)}{(x-y)q(x, y)} \frac{dx \wedge d_\pm^c x}{A(x)} \\ &\quad + \left(B'(y) + \frac{q(y) - q_0 (x-y)^2}{(x-y)q(x, y)} B(y) \right) \frac{dy \wedge d_\pm^c y}{B(y)}. \end{aligned}$$

Hence for any \mathfrak{t} -invariant function $\phi = \phi(x, y)$,

$$\begin{aligned} dd_\pm^c \phi &= \phi_{xx} dx \wedge d_\pm^c x + \phi_{yy} dy \wedge d_\pm^c y + \phi_{xy} (dx \wedge d_\pm^c y + dy \wedge d_\pm^c x) \\ &\quad + \phi_x dd_\pm^c x + \phi_y dd_\pm^c y \end{aligned}$$

$$\begin{aligned}
 &= \left((A(x)\phi_x)_x - \frac{q(x) - q_0(x-y)^2}{(x-y)q(x,y)} A(x)\phi_x \right) \frac{dx \wedge d_{\pm}^c x}{A(x)} \\
 &\quad \pm \frac{q(y)A(x)\phi_x}{(x-y)q(x,y)} \frac{dy \wedge d_{\pm}^c y}{B(y)} \mp \frac{q(x)B(y)\phi_y}{(x-y)q(x,y)} \frac{dx \wedge d_{\pm}^c x}{A(x)} \\
 &\quad + \left((B(y)\phi_y)_y + \frac{q(y) - q_0(x-y)^2}{(x-y)q(x,y)} B(y)\phi_y \right) \frac{dy \wedge d_{\pm}^c y}{B(y)} \\
 &\quad + \phi_{xy}(dx \wedge d_{\pm}^c y + dy \wedge d_{\pm}^c x).
 \end{aligned}$$

In particular, the expression is both J_+ and J_- invariant if and only if $\phi_{xy} = 0$. The invariant part simplifies considerably when expressed in terms of the Kähler forms ω_{\pm}^0 of the barycentric metric. Using the fact that $q_0x + q_1$ and $q_0y + q_1$ are the y and x derivatives of $q(x, y)$ respectively, we eventually obtain

$$\begin{aligned}
 dd_{\pm}^c \phi &= \frac{q(x, y)^2}{2} \left(\left[\frac{A(x)\phi_x}{q(x, y)^2} \right]_x \pm \left[\frac{B(y)\phi_y}{q(x, y)^2} \right]_y \right) \omega_{\pm}^0 \\
 &\quad + \frac{(x-y)^2}{2} \left(\left[\frac{A(x)\phi_x}{(x-y)^2} \right]_x \mp \left[\frac{B(y)\phi_y}{(x-y)^2} \right]_y \right) \omega_{\mp}^0 \\
 &\quad + \phi_{xy}(dx \wedge d_{\pm}^c y + dy \wedge d_{\pm}^c x).
 \end{aligned}$$

Substituting the Ricci potentials for ϕ , we thus obtain, after a little manipulation,

$$\begin{aligned}
 \rho_+ &= -\frac{q(x, y)^2}{4} \left(\left[q(x, y)^2 \left[\frac{A(x)}{q(x, y)^4} \right]_x \right]_x + \left[q(x, y)^2 \left[\frac{B(y)}{q(x, y)^4} \right]_y \right]_y \right) \omega_+^0 \\
 &\quad - \frac{(x-y)^2}{4} \left(\left[\frac{q(x, y)^4}{(x-y)^2} \left[\frac{A(x)}{q(x, y)^4} \right]_x \right]_x - \left[\frac{q(x, y)^4}{(x-y)^2} \left[\frac{B(y)}{q(x, y)^4} \right]_y \right]_y \right) \omega_-^0, \\
 &\quad + 2 \frac{(q_0q_2 - q_1^2)(dx \wedge d_+^c y + dy \wedge d_+^c x)}{q(x, y)^2} \\
 \rho_- &= -\frac{q(x, y)^2}{4} \left(\left[\frac{(x-y)^4}{q(x, y)^2} \left[\frac{A(x)}{(x-y)^4} \right]_x \right]_x - \left[\frac{(x-y)^4}{q(x, y)^2} \left[\frac{B(y)}{(x-y)^4} \right]_y \right]_y \right) \omega_+^0 \\
 &\quad - \frac{(x-y)^2}{4} \left(\left[(x-y)^2 \left[\frac{A(x)}{(x-y)^4} \right]_x \right]_x + \left[(x-y)^2 \left[\frac{B(y)}{(x-y)^4} \right]_y \right]_y \right) \omega_-^0 \\
 &\quad + 2 \frac{dx \wedge d_-^c y + dy \wedge d_-^c x}{(x-y)^2}.
 \end{aligned}$$

(In particular g_+ can only be Kähler–Einstein in the parabolic case—when q has a repeated root—while g_- is never Kähler–Einstein.) The scalar curvatures are given by $s_{\pm} = 2\rho_{\pm} \wedge \omega_{\pm}/v_{\pm}$. Expanding the derivatives, we then obtain

$$\begin{aligned}
 (33) \quad s_+ &= \frac{1}{(x-y)q(x, y)} \left(-q(x, y)^2(A''(x) + B''(y)) \right. \\
 &\quad \left. + 6q(x, y)((q_0y + q_1)A'(x) + (q_0x + q_1)B'(y)) \right. \\
 &\quad \left. - 12((q_0y + q_1)^2A(x) + (q_0x + q_1)^2B(y)) \right),
 \end{aligned}$$

$$\begin{aligned}
 (34) \quad s_- &= \frac{1}{(x-y)q(x, y)} \left(-(x-y)^2(A''(x) + B''(y)) \right. \\
 &\quad \left. + 6(x-y)(A'(x) - B'(y)) - 12(A(x) + B(y)) \right).
 \end{aligned}$$

6.2. Extremality and Bach-flatness. The Kähler metrics g_{\pm} are extremal if their scalar curvatures s_{\pm} are Killing potentials. Since the latter are \mathfrak{t} -invariant (and \mathfrak{t} is lagrangian), this can only happen if s_{\pm} is the momentum of some Killing vector field $K^{(p)} \in \mathfrak{t}$. The condition is straightforward to solve for g_+ : equating (33) (for s_+) and (32) (for $\mu_+^{(p)}$) yields

$$(35) \quad -q(x, y)^2(A''(x) + B''(y)) + 3q(x, y)(q'(y)A'(x) + q'(x)B'(y)) \\ - 3(q'(y)^2A(x) + q'(x)^2B(y)) = (x - y)(w_0xy + w_1(x + y) + w_2).$$

Differentiating three times with respect to x or three times with respect to y shows that A and B (respectively) are polynomials of degree at most four. We now introduce polynomials Π and P determined by $A = \Pi + P$ and $B = \Pi - P$. Since the left hand side of (35) is antisymmetric in (x, y) , the symmetric part of the equation is

$$-q(x, y)^2(\Pi''(x) + \Pi''(y)) + 3q(x, y)(q'(y)\Pi'(x) + q'(x)\Pi'(y)) \\ - 3(q'(y)^2\Pi(x) + q'(x)^2\Pi(y)) = 0.$$

On restriction to the diagonal ($x = y$) in this polynomial equation, we obtain

$$q^2\Pi'' - 3qq'\Pi' + 3(q')^2\Pi = 0.$$

To solve this linear ODE for Π , we set $\Pi(z) = q(z)\pi(z)$ to get $q^2(q''\pi - q'\pi' + q\pi'') = 0$, from which we deduce that π is a polynomial of degree ≤ 2 ($\pi''' = 0$) and that π is orthogonal to q .

The antisymmetric part of (35) is

$$-q(x, y)^2(P''(x) - P''(y)) + 3q(x, y)(q'(y)P'(x) - q'(x)P'(y)) \\ - 3(q'(y)^2P(x) - q'(x)^2P(y)) = (x - y)(w_0xy + w_1(x + y) + w_2).$$

The left hand side is clearly divisible by $x - y$; it is less clear that the quotient by $(x - y)q(x, y)$ is a Killing potential, but this can be checked by explicit computation or by considering normal forms for q (see below). To compute this Killing potential, we divide the left hand side by $x - y$ and restrict to the diagonal to obtain

$$-q^2P''' + 3qq'P'' - 3((q')^2 + qq'')P' + 6q'q''P \\ = q'(qP'' - 3q'P' + 6q''P) - q(qP'' - 3q'P' + 6q''P)'. \quad (36)$$

Since P has degree ≤ 4 , and q has degree ≤ 2 , it follows (for instance by differentiating three times) that

$$(36) \quad q \cdot P := qP'' - 3q'P' + 6q''P$$

has degree ≤ 2 (this expression is an example of a transvectant for $\text{SL}(2)$). As q is nonzero, any quadratic form may be represented as $q \cdot P$ for some quartic P . However, the Wronskian $\{q, q \cdot P\} := q'(q \cdot P) - q(q \cdot P)'$ (which is also, in homogeneous coordinates, the Poisson bracket on quadratic forms) is automatically orthogonal to q . Any such quadratic form arises from a quartic P in this way. Thus

$$s_+ = \frac{w(x, y)}{q(x, y)},$$

where $w(x, y) = w_0xy + w_1(x + y) + w_2$ is the polarization of a quadratic form $w(z) = w(z, z)$ depending linearly on P and quadratically on q , and which is orthogonal to q . Hence, except in the parabolic case (q degenerate), s_+ is constant if and only if it is identically zero.

Remarkably, the extremality condition for g_- coincides with that for g_+ . To see this, we equate (34) (for s_-) and (31) (for $\mu_-^{(p)}$, with an arbitrary constant of integration) to obtain the extremality equation

$$(37) \quad -(x-y)^2(A''(x) + B''(y)) + 6(x-y)(A'(x) - B'(y)) - 12(A(x) + B(y)) \\ = q(x, y)(p_0xy + p_1(x+y) + p_2 + c(x-y)),$$

which we shall again decompose into symmetric and antisymmetric parts: for this we first observe, by taking three derivatives, that A and B are polynomials of degree ≤ 4 , we write $A = \Pi + P$, $B = \Pi - P$ as before.

The symmetric part, namely

$$-(x-y)^2(\Pi''(x) + \Pi''(y)) + 6(x-y)(\Pi'(x) - \Pi'(y)) - 12(\Pi(x) + \Pi(y)) \\ = q(x, y)(p_0xy + p_1(x+y) + p_2),$$

immediately yields, on restricting to the diagonal, $\Pi(z) = q(z)\pi(z)$ with $\pi(z) = -p(z)/24$. Further, the equation is satisfied with this Ansatz.

The antisymmetric part, namely

$$-(x-y)^2(P''(x) - P''(y)) + 6(x-y)(P'(x) + P'(y)) - 12(P(x) - P(y)) \\ = cq(x, y)(x-y)$$

yields $c = 0$ (divide by $x-y$ and restrict to the diagonal) and is then satisfied identically for *any* polynomial P of degree ≤ 4 . Thus we again have an extremal Kähler metric with

$$s_- = -\frac{24\pi(x, y)}{x-y},$$

where $\pi(x, y) = \pi_0xy + \pi_1(x+y) + \pi_2$ is the polarization of $\pi(z) = \pi(z, z)$. Note that s_- is constant if and only if it is identically zero.

The Bach-flat condition is readily found using Lemma 2: since $-v_-/v_+ = q(x, y)^4/(x-y)^4$, equation (6) holds if and only if $\pi(x, y)$ and $w(x, y)$ are linearly dependent.

Theorem 3. *Let $(J_+, J_-, g_+, g_-, \mathfrak{t})$ be a generic ambitoric structure as in Theorem 2. Then (g_+, J_+) is an extremal Kähler metric if and only if (g_-, J_-) is an extremal Kähler metric if and only if*

$$(38) \quad A(z) = q(z)\pi(z) + P(z), \\ B(z) = q(z)\pi(z) - P(z),$$

where $\pi(z)$ is a polynomial of degree at most two orthogonal to $q(z)$ and $P(z)$ is polynomial of degree at most four. The conformal structure is Bach-flat if and only if the quadratic polynomials π and $\{q, q \cdot P\}$ (described above) are linearly dependent.

In view of Theorem 1, we thus obtain a local description of Einstein 4-manifolds with degenerate half-Weyl tensors.

Corollary 1. *Let (M, g) be a Einstein 4-manifold for which the half-Weyl tensors W^+ and W^- are everywhere degenerate. Then on dense open subset of M , the metric g is locally homothetic to one of the following:*

- a real space form;
- a product of two Riemann surfaces with equal constant Gauss curvatures;
- an Einstein metric of the form $s_+^{-2}g_+$, where g_+ is a Bach-flat Kähler metric with nonvanishing scalar curvature s_+ , described in Proposition 5 or Theorem 3.

6.3. Normal forms and summary.

Definition 10. A regular ambitoric conformal structure is said to be of *elliptic*, *parabolic*, or *hyperbolic* type if the number of distinct real roots of $q(z)$ (on $\mathbb{P}(W)$) is zero, one or two respectively.

In these three cases we use the projective freedom to set $q(z) = z$, $q(z) = 1$ and $q(z) = 1 + z^2$ respectively. For later reference, we now consider each of the three types in turn.

Parabolic type. We first consider the case that $q(z) = 1$ when the barycentric metric g_0 and Kähler forms ω_{\pm} are as follows:

$$\begin{aligned} g_0 &= \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + \frac{A(x)(dt_1 + y dt_2)^2}{(x - y)^2} + \frac{B(y)(dt_1 + x dt_2)^2}{(x - y)^2}, \\ \omega_+ &= dx \wedge (dt_1 + y dt_2) + dy \wedge (dt_1 + x dt_2), \\ \omega_- &= \frac{dx \wedge (dt_1 + y dt_2)}{(x - y)^2} - \frac{dy \wedge (dt_1 + x dt_2)}{(x - y)^2}. \end{aligned}$$

The Killing potentials (or momenta) of the Killing vector fields K_1 and K_2 dual to dt_1 and dt_2 are given by

$$(39) \quad \mu_1^+ = x + y, \quad \mu_2^+ = xy,$$

$$(40) \quad \mu_1^- = -\frac{1}{x - y}, \quad \mu_2^- = -\frac{x + y}{2(x - y)}.$$

The metric g_+ is extremal if and only if g_- is extremal if and only if $A(z) = a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4$, $B(z) = b_0 z^4 + b_1 z^3 + b_2 z^2 + b_3 z + b_4$ and

$$(41) \quad a_0 + b_0 = a_1 + b_1 = a_2 + b_2 = 0.$$

In this case

$$\begin{aligned} s_+ &= -6a_1 - 12a_0 \mu_1^+ \\ s_- &= 12(a_4 + b_4) \mu_1^- + 12(a_3 + b_3) \mu_2^-. \end{aligned}$$

The Bach-flat condition is that $a_1 + 4a_0 z$ and $-(a_4 + b_4) + (a_3 + b_3)z$ are linearly dependent, i.e.,

$$(42) \quad a_1(a_3 + b_3) + 4a_0(a_4 + b_4) = 0.$$

Hyperbolic type. We next consider the case that $q(z) = z$ when the barycentric metric g_0 and Kähler forms ω_{\pm} are as follows:

$$\begin{aligned} g_0 &= \frac{dx^2}{A(x)} + \frac{dy^2}{A(y)} + \frac{A(x)(dt_1 + y^2 dt_2)^2}{(x^2 - y^2)^2} + \frac{B(y)(dt_1 + x^2 dt_2)^2}{(x^2 - y^2)^2} \\ \omega_+ &= \frac{dx \wedge (dt_1 + y^2 dt_2)}{(x + y)^2} + \frac{dy \wedge (dt_1 + x^2 dt_2)}{(x + y)^2} \\ \omega_- &= \frac{dx \wedge (dt_1 + y^2 dt_2)}{(x - y)^2} - \frac{dy \wedge (dt_1 + x^2 dt_2)}{(x - y)^2} \end{aligned}$$

The momenta of K_1 and K_2 are given by

$$(43) \quad \mu_1^+ = -\frac{1}{x + y}, \quad \mu_2^+ = \frac{xy}{x + y},$$

$$(44) \quad \mu_1^- = -\frac{1}{x - y}, \quad \mu_2^- = -\frac{xy}{x - y}.$$

Again, g_+ is extremal if and only if g_- is extremal if and only if $A(z) = a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4$, $B(z) = b_0z^4 + b_1z^3 + b_2z^2 + b_3z + b_4$ and

$$(45) \quad a_0 + b_0 = a_2 + b_2 = a_4 + b_4 = 0.$$

In this case the scalar curvatures are

$$(46) \quad s_{\pm} = -6(a_3 \pm b_3)\mu_1^{\pm} - 6(a_1 \pm b_1)\mu_2^{\pm}.$$

The condition corresponding to the vanishing of the Bach tensor is therefore

$$(47) \quad (a_3 - b_3)(a_1 + b_1) + (a_3 + b_3)(a_1 - b_1) = 0.$$

Elliptic type. We finally consider the case that $q(z) = 1 + z^2$ where the barycentric metric g_0 and Kähler forms ω_{\pm} are as follows:

$$\begin{aligned} g_0 &= \frac{dx^2}{A(x)} + \frac{dy^2}{B(y)} + \frac{A(x)(dt_1 + (y^2 - 1)dt_2)^2}{(x - y)^2(1 + xy)^2} + \frac{B(y)(dt_1 + (x^2 - 1)dt_2)^2}{(x - y)^2(1 + xy)^2} \\ \omega_+ &= \frac{dx \wedge (2y dt_1 + (y^2 - 1)dt_2)}{(1 + xy)^2} + \frac{dy \wedge (2x dt_1 + (x^2 - 1)dt_2)}{(1 + xy)^2} \\ \omega_- &= \frac{dx \wedge (2y dt_1 + (y^2 - 1)dt_2)}{(x - y)^2} - \frac{dy \wedge (2x dt_1 + (x^2 - 1)dt_2)}{(x - y)^2} \end{aligned}$$

with momenta of K_1 and K_2 equal to

$$(48) \quad \mu_1^+ = -\frac{1 - xy}{1 + xy}, \quad \mu_2^+ = -\frac{x + y}{1 + xy},$$

$$(49) \quad \mu_1^- = -\frac{x + y}{x - y}, \quad \mu_2^- = \frac{1 - xy}{x - y}.$$

The extremal case is now $A(z) = a_0z^4 + a_1z^3 + a_2z^2 + a_3z + a_4$, $B(z) = b_0z^4 + b_1z^3 + b_2z^2 + b_3z + b_4$ with

$$(50) \quad a_2 + b_2 = 0, \quad a_0 + b_0 + a_4 + b_4 = 0, \quad a_1 + b_1 = a_3 + b_3$$

in which case the scalar curvatures are

$$(51) \quad s_+ = 6(a_3 - b_1)\mu_1^+ - 12(a_4 + b_0)\mu_2^+.$$

$$(52) \quad s_- = 12(a_3 + b_3)\mu_1^- + 12(a_4 + b_4)\mu_2^-.$$

The Bach-flatness condition is therefore:

$$(53) \quad (a_3 - b_1)(a_3 + b_3) + 4(a_4 + b_4)(a_4 + b_0) = 0$$

Summary table. The following table summarizes the above computations.

Condition	Parabolic type	Hyperbolic type	Elliptic type
g_{\pm} extremal	$a_0 + b_0 = 0$ $a_1 + b_1 = 0$ $a_2 + b_2 = 0$	$a_0 + b_0 = 0$ $a_2 + b_2 = 0$ $a_4 + b_4 = 0$	$a_0 + b_0 + a_4 + b_4 = 0$ $a_2 + b_2 = 0$ $a_1 + b_1 = a_3 + b_3$
g_{\pm} Bach-flat	extremal and $a_1(a_3 + b_3) = -4a_0(a_4 + b_4)$	extremal and $(a_3 - b_3)(a_1 + b_1) = -(a_3 + b_3)(a_1 - b_1)$	extremal and $(a_3 - b_1)(a_3 + b_3) = -4(a_4 + b_4)(a_4 + b_0)$
$s_+ \equiv 0$ ($W_+ \equiv 0$)	extremal and $a_0 = 0$ $a_1 = 0$	extremal and $a_1 = b_1$ $a_3 = b_3$	extremal and $a_3 = b_1$ $a_4 = -b_0$
$s_- \equiv 0$ ($W_- \equiv 0$)	extremal and $a_3 = -b_3$ $a_4 = -b_4$	extremal and $a_1 = -b_1$ $a_3 = -b_3$	extremal and $a_3 = -b_3$ $a_4 = -b_4$

g_- is never Kähler–Einstein, and has constant scalar curvature iff $s_- \equiv 0$. The same holds for g_+ except in the parabolic case, when g_+ has constant scalar curvature iff it is extremal with $a_0 = 0$, and is Kähler–Einstein if also $a_3 + b_3 = 0$.

7. AMBITORIC POLYTOPES AND AMBITORIC COMPACTIFICATIONS

7.1. Toric manifolds and orbifolds. We begin by reviewing the theory of toric Kähler manifolds and orbifolds, primarily adopting the symplectic point of view as in [2, 21, 30, 31, 38]. Our treatment follows closely the exposition in [8, §1]: in particular we use first order boundary conditions for the compactification of compatible Kähler metrics on toric symplectic orbifolds. Also, we use basis independent language for the lattice in \mathfrak{t} defining the torus \mathbb{T} , since the bases arising in examples are not necessarily compatible with this lattice.

Let \mathfrak{t} be an m -dimensional real vector space. Recall that a *rational Delzant polytope* $(\Delta, \Lambda, u_1, \dots, u_n)$ in \mathfrak{t}^* is a compact convex polytope $\Delta \subset \mathfrak{t}^*$ equipped with *normals* belonging to a lattice Λ in \mathfrak{t}

$$(54) \quad u_j \in \Lambda \subset \mathfrak{t}$$

($j = 1, \dots, n$, $n > m$) such that

$$(55) \quad \Delta = \{\mu \in \mathfrak{t}^* : L_j(\mu) \geq 0, j = 1, \dots, n\}$$

with

$$L_j(\mu) = \langle u_j, \mu \rangle + \lambda_j$$

for some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, and such that for any vertex $\mu \in \Delta$, the u_j with $L_j(\mu) = 0$ form a basis for \mathfrak{t} . If the normals form a basis for Λ at each vertex, then Δ is said to be *integral*, or simply a *Delzant polytope*.

The term *rational* refers to the fact that the normals span an m -dimensional vector space over \mathbb{Q} . A rational Delzant polytope is obviously m -valent, i.e., m codimension one faces and m edges meet at each vertex: by (55) the codimension one faces F_1, \dots, F_n are given by $F_j = \Delta \cap \{\mu \in \mathfrak{t}^* : L_j(\mu) = 0\}$, so that u_j is an inward normal vector to F_j . In the integral case, the u_j are necessarily primitive, and so are uniquely determined by (Δ, Λ) . In general, the primitive inward normals are u_j/m_j for some positive integer labelling m_j of the codimension one faces F_j , so rational Delzant polytopes are also called *labelled polytopes* [38]. However, it turns out to be more convenient to encode the labelling in the normals. Note that $\lambda_1, \dots, \lambda_n$ are uniquely determined by $(\Delta, u_1, \dots, u_n)$.

A connected $2m$ -dimensional symplectic orbifold (M, ω) is *toric* if it is equipped with an effective hamiltonian action of an m -torus \mathbb{T} with momentum map $\mu: M \rightarrow \mathfrak{t}^*$. The rational Delzant theorem [21, 38] states that compact toric symplectic orbifolds are classified (up to equivariant symplectomorphism) by rational Delzant polytopes (with manifolds corresponding to integral Delzant polytopes): given such a polytope, (M, ω) is obtained as a symplectic quotient of \mathbb{C}^n by an $(n - m)$ -dimensional subgroup G of the standard n -torus $(S^1)^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$: precisely, G is the kernel of the map $(S^1)^n \rightarrow \mathbb{T} = \mathfrak{t}/2\pi\Lambda$ induced by the map $(x_1, \dots, x_n) \mapsto \sum_{j=1}^n x_j u_j$ from \mathbb{R}^n to \mathfrak{t} , and the momentum level for the symplectic quotient is the image in \mathfrak{t}^* of $(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^{n*}$ under the transpose of the natural inclusion of the Lie algebra \mathfrak{t} in \mathbb{R}^n . Conversely, a toric symplectic orbifold gives rise to a rational Delzant polytope as the image Δ of its momentum map μ , where Λ is the lattice of circle subgroups, and the positive integer labelling m_j of the codimension one faces F_j is determined by the fact that the local uniformizing group of every point in $\mu^{-1}(F_j^0)$ is $\mathbb{Z}/m_j\mathbb{Z}$. (For any face F , we denote by F^0 its interior.)

A compact convex polytope with chosen inward normals (giving a basis for \mathfrak{t} at each vertex) is a rational Delzant polytope with respect to *any* lattice satisfying (54). There is clearly there is a smallest such lattice Λ , generated by the normals, and any other such lattice Λ' is a sublattice (of finite index) in this. The torus $\mathbb{T}' = \mathfrak{t}/2\pi\Lambda'$ is the quotient of $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$ by a finite abelian group $\Gamma \cong \Lambda'/\Lambda$, and the corresponding toric symplectic orbifolds M and M' (under the tori \mathbb{T} and \mathbb{T}') are related by a regular orbifold covering [44]: $M' = M/\Gamma$. In fact M is a *simply connected orbifold* in the sense of Thurston [44] and is the universal orbifold cover of M' . We can thus focus, with no essential loss of generality, on simply connected toric orbifolds, whose polytopes may be described as follows.

Definition 11. A *simply connected* rational Delzant polytope is a compact convex polytope $\Delta \subset \mathfrak{t}^*$ equipped with inward normals $u_1, \dots, u_n \in \mathfrak{t}$ such that $\Lambda := \text{span}_{\mathbb{Z}}\{u_1, \dots, u_n\}$ is a lattice of \mathfrak{t} , and for any vertex the normals corresponding to the adjacent facets form a \mathbb{Q} -basis for \mathfrak{t} .

We turn now to the study of compatible Kähler metrics on toric symplectic orbifolds. On the union $M^0 := \mu^{-1}(\Delta^0)$ of the generic orbits, such metrics have an explicit description due to Guillemin [30, 31]. Orthogonal to the orbits is a rank m distribution spanned by commuting holomorphic vector fields JX_ξ for $\xi \in \mathfrak{t}$. Hence there is a function $t: M^0 \rightarrow \mathfrak{t}/2\pi\Lambda$, defined up to an additive constant, such that $dt(JX_\xi) = 0$ and $dt(X_\xi) = \xi$ for $\xi \in \mathfrak{t}$. The components of t are ‘angular variables’, complementary to the components of the momentum map $\mu: M^0 \rightarrow \mathfrak{t}^*$, and the symplectic form in these coordinates is simply $\omega = \langle d\mu \wedge dt \rangle$, where the angle brackets denote contraction of \mathfrak{t} and \mathfrak{t}^* .

These coordinates identify each tangent space with $\mathfrak{t} \oplus \mathfrak{t}^*$, so any \mathbb{T} -invariant ω -compatible Kähler metric must be of the form

$$(56) \quad g = \langle d\mu, \mathbf{G}, d\mu \rangle + \langle dt, \mathbf{H}, dt \rangle,$$

where \mathbf{G} is a positive definite $S^2\mathfrak{t}$ -valued function on Δ^0 , \mathbf{H} is its inverse in $S^2\mathfrak{t}^*$ —observe that \mathbf{G} and \mathbf{H} define mutually inverse linear maps $\mathfrak{t}^* \rightarrow \mathfrak{t}$ and $\mathfrak{t} \rightarrow \mathfrak{t}^*$ at each point—and $\langle \cdot, \cdot, \cdot \rangle$ denotes the pointwise contraction $\mathfrak{t}^* \times S^2\mathfrak{t} \times \mathfrak{t}^* \rightarrow \mathbb{R}$ or the dual contraction. The corresponding almost complex structure is defined by

$$(57) \quad Jdt = -\langle \mathbf{G}, d\mu \rangle$$

from which it follows that J is integrable if and only if \mathbf{G} is the Hessian of a function (called symplectic potential) on Δ^0 [30].

Necessary and sufficient conditions for \mathbf{G} to come from a globally defined \mathbb{T} -invariant ω -compatible Kähler metric on M are obtained in [2, 8, 24]. In order to state the criterion from [8], we introduce some notation. For any face $F \subset \Delta$, we denote by $\mathfrak{t}_F \subset \mathfrak{t}$ the vector subspace spanned by the inward normals $u_j \in \mathfrak{t}$ to all codimension one faces of Δ , containing F ; thus the codimension of \mathfrak{t}_F equals the dimension of F . Furthermore, the annihilator \mathfrak{t}_F^0 of \mathfrak{t}_F in \mathfrak{t}^* is naturally identified with $(\mathfrak{t}/\mathfrak{t}_F)^*$.

Proposition 9. [8] *Let (M, ω) be a compact toric symplectic $2m$ -manifold or orbifold with momentum map $\mu: M \rightarrow \Delta \subset \mathfrak{t}^*$, and \mathbf{H} be a positive definite $S^2\mathfrak{t}^*$ -valued function on Δ^0 . Then \mathbf{H} defines a \mathbb{T} -invariant, ω -compatible almost Kähler metric g via (56) if and only if it satisfies the following conditions:*

- [smoothness] \mathbf{H} is the restriction to Δ^0 of a smooth $S^2\mathfrak{t}^*$ -valued function on Δ ;
- [boundary values] for any point μ on the codimension one face $F_j \subset \Delta$ with inward normal u_j , we have

$$(58) \quad \mathbf{H}_\mu(u_j, \cdot) = 0 \quad \text{and} \quad (d\mathbf{H})_\mu(u_j, u_j) = 2u_j,$$

- where the differential $d\mathbf{H}$ is viewed as a smooth $S^2\mathfrak{t}^* \otimes \mathfrak{t}$ -valued function on Δ ;
- [positivity] for any point μ in interior of a face $F \subseteq \Delta$, $\mathbf{H}_\mu(\cdot, \cdot)$ is positive definite when viewed as a smooth function with values in $S^2(\mathfrak{t}/\mathfrak{t}_F)^*$.

7.2. Orbifold compactifications of ambitoric Kähler surfaces. In order to compactify the Kähler metrics g_\pm introduced in section 6, we use the respective momentum coordinates $\mu = (\mu_1^\pm, \mu_2^\pm)$ to rewrite the Kähler metrics in the setting of section 7.1.⁴ The hamiltonian Killing fields K_1, K_2 corresponding to the these momentum coordinates will not in general be S^1 generators for the \mathbb{T} action.

Definition 12. An *ambitoric compactification* is a simply connected, compact ambikähler 4-orbifold $(M, J_\pm, g_\pm, \omega_\pm)$ endowed with an effective hamiltonian action of a 2-torus \mathbb{T} by isometries. Since the union M^0 of generic orbits of the \mathbb{T} action is open and dense, M is ambitoric in the sense of Definition 3.

As a preliminary remark, we observe that ambitoric compactifications are always singular, unless M is a complex Hirzebruch surface, in which case (g_+, J_+, ω_+) can be taken to be of Calabi type.

Proposition 10. Suppose that (M, c, J_+, J_-) is a nonsingular ambitoric compactification. Then, (M, J_+) (and (M, J_-)) is biholomorphic to a Hirzebruch surface $F_k = P(\mathcal{O} \oplus \mathcal{O}(k)) \rightarrow \mathbb{CP}^1$, $k \in \mathbb{N}$.

Conversely, each F_k admits compatible ambikähler structures (c, J_+, J_-) which are toric with respect to a maximal torus \mathbb{T} in the automorphism group, and are ambitoric of Calabi type on the dense open subset of points where \mathbb{T} acts freely. Any ambikähler structure (c, J_+, J_-) on F_k for which the conformal Kähler metrics g_\pm are extremal is of this type.

Proof. A compact Kähler surface admitting a holomorphic and hamiltonian torus action is rational, and therefore has nonpositive signature, unless it is \mathbb{CP}^2 . However, \mathbb{CP}^2 doesn't admit an almost complex structure inducing the nonstandard orientation, so that both (M, J_+) and (M, J_-) are rational surfaces with nonnegative signature. Since J_+ and J_- yield opposite orientations on the M , the signature of M must be zero. The rational complex surfaces with zero signature are the Hirzebruch surfaces [10]. Not only that they are all toric, but they do admit toric extremal metrics of Calabi type in each Kähler class [13]. These are ambitoric on the regular points of the torus action. If (c, J_+, J_-) is ambitoric with g_\pm extremal, by the uniqueness of the extremal Kähler metrics in their Kähler classes, we can assume that (g_+, J_+) is given by the Calabi construction. As such a metric is already ambikähler with respect to a negative complex structure \tilde{J}_- and is not self-dual, J_- must be $\pm\tilde{J}_-$ ([22, 4]). Thus, the ambikähler structure (c, J_+, J_-) is ambitoric of Calabi type on a dense open subset of F_k . \square

In view of the above result and our local classification in Theorem 2, we will discuss from now on (singular) toric orbifold compactifications of the three types of regular ambitoric Kähler metrics.

Assumption 1. We will assume that the (x, y) -coordinates are well-defined on the dense open subset M^0 of points where the \mathbb{T}^2 -action is free (i.e., on the pre-image of the interior of the rational Delzant polytope Δ^+ associated to ω_+).

We know from Proposition 9 that the determinant of the matrix $\mathbf{H} = (H_{ij})$ (where $H_{ij} = g_+(K_i, K_j)$) must be smooth on M and vanish on (the pre-image

⁴As we are interested in 'globally' ambikähler structures, it is enough to consider one of the metrics, (g_+, ω_+) say.

of) the boundary of the Delzant polytope. A case-by-case verification shows that $\det \mathbf{H}$ is a nonzero multiple of $A(x)B(y)$ (taking also in mind that we want both g_{\pm} to extend on M). It follows that (x, y) must be defined on a domain $D^0 := (\alpha_1, \alpha_2) \times (\beta_1, \beta_2)$ with $\beta_2 \leq \alpha_1$, where α_i (resp. β_j) are zeroes of $A(z)$ (resp. $B(z)$) such that $A(z)$ (resp. $B(z)$) is strictly positive on (α_1, α_2) (resp. (β_1, β_2)). The main point here is that formulae (39)–(40), (43)–(44) and (48)–(49) in turn show that the closure D of D^0 transforms into a compact convex polytope, denoted in the sequel by Δ^+ . Note that Δ^+ has four facets, unless $\beta_2 = \alpha_1$ when it is a simplex.

Remark 4. (Weighted projective spaces.) The case when Δ^+ is a simplex has been studied in detail in [12, 2, 8]. Simply connected rational Delzant simplices parameterize weighted projective spaces. Any such orbifold has a unique Kähler class up to homothety, and a unique extremal Kähler metric which is Bochner-flat. In four-dimensions, this is the case corresponding to toric compactifications of an extremal Kähler metric g_+ with $W_- = 0$ (or equivalently $s_- = 0$). We thus obtain different ambitoric realizations of the (same) selfdual metric, due to the fact that there are many choices for the negative complex structure J_- , a situation that can only occur (even locally) when $W_- = 0$ (see e.g., [4]). However, none of the complex structures J_- is defined on the whole of M , so that a weighted projective space is locally, but not globally ambitoric⁵.

This remark motivates a further assumption.

Assumption 2. We suppose from now on that $\beta_2 < \alpha_1$, so that the polytope Δ^+ has 4 facets and the momentum transformation is nondegenerate in a neighbourhood of D^0 . By Assumption 1, in this case the smooth functions (x, y) are defined everywhere on M .

We now proceed by a case-by-case analysis of the three types of regular ambitoric Kähler metrics, by using the normal forms obtained in section 6.3.

7.2.1. Parabolic type. Under Assumption 1, this is the case when (J_+, g_+, ω_+) is orthotoric in the sense of [6]. A description of compact orthotoric Kähler orbifolds (of any dimension) is given in [8] and their complete classification in the 4-dimensional case appears in [37]. For completeness, we recall the construction from [8], specialized to the 4-dimensional case.

From formulae (39)–(40), we see that in this case the polytope Δ^+ is

$$\begin{aligned} \Delta^+ = \{ & (\mu_1^+, \mu_2^+) : L_j^\alpha(\mu_1^+, \mu_2^+) \geq 0; L_j^\beta(\mu_1^+, \mu_2^+) \geq 0, j = 1, 2 \}, \\ (59) \quad & L_j^\alpha(\mu_1^+, \mu_2^+) = (-1)^j (-\alpha_j \mu_1^+ + \mu_2^+ + \alpha_j^2), \\ & L_j^\beta(\mu_1^+, \mu_2^+) = (-1)^{j+1} (-\beta_j \mu_1^+ + \mu_2^+ + \beta_j^2). \end{aligned}$$

The inward normals are then of the form

$$(60) \quad u_j^\alpha = c_j^\alpha(\alpha_j, -1), \quad u_j^\beta = c_j^\beta(\beta_j, -1), \quad j = 1, 2,$$

for some real numbers c_j^α, c_j^β satisfying $c_1^\alpha > 0 > c_2^\alpha$, $c_1^\beta < 0 < c_2^\beta$. The condition that $(\Delta^+, u_j^\alpha, u_j^\beta)$ is a simply connected rational Delzant polytope reduces to

$$(61) \quad \text{span}_{\mathbb{Z}}\{u_1^\alpha, u_2^\alpha, u_1^\beta, u_2^\beta\} \cong \mathbb{Z}^2,$$

which imposes an integrality condition on the real numbers $(\alpha_j, \beta_j, c_j^\alpha, c_j^\beta)$.

⁵The formula (30) for the conformal factor f shows that if $x = y$ at some points of M , then the toric metrics g_+ and g_- cannot be simultaneously compactified

Remark 5. A particular case where (61) holds automatically is when all the numbers α_j, β_j and c_j^α, c_j^β are rational: since the condition (61) is clearly invariant under an overall multiplication of c_j^α and c_j^β by a nonzero real constant, we can choose this constant such that u_j^α and u_j^β have integer coordinates.

By formula (30) (specialized to $q(z) = 1$) we have $g_- = \frac{1}{(x-y)^2} g_+$. As $\alpha_1 > \beta_2$, it follows that g_- compactifies if and only if g_+ does.

We now impose the boundary conditions of Proposition 9.

We work on D , rather than on Δ^+ , using the fact that the transformation (39)–(40) is nondegenerate and sends a facet of Δ^+ to a facet of D . We find that

$$(62) \quad \mathbf{H} = \frac{1}{x-y} \begin{pmatrix} A(x) + B(y) & y A(x) + x B(y) \\ y A(x) + x B(y) & y^2 A(x) + x^2 B(y) \end{pmatrix},$$

which, together with the formulae (39)–(40), shows that \mathbf{H} is smooth as soon as $A(z)$ and $B(z)$ are smooth functions on $[\alpha_1, \alpha_2]$ and $[\beta_1, \beta_2]$, respectively. Also, the positivity condition is equivalent to $A > 0$ on (α_1, α_2) and $B > 0$ on (β_1, β_2) .

The first boundary value condition, $\mathbf{H}(u_j, \cdot) = 0$ on $\partial\Delta^+$, reduces to $A(\alpha_1) = A(\alpha_2) = B(\beta_1) = B(\beta_2) = 0$. The second boundary value condition, $d\mathbf{H}(u_j, u_j) = 2u_j$ on $\partial\Delta^+$, has an invariant sense on D by considering the normals u_j^α and u_j^β as 1-forms on Δ^+ and on D , as follows:

$$\begin{aligned} u_j^\alpha &= c_j^\alpha (\alpha_j d\mu_1^+ - d\mu_2^+) = c_j^\alpha ((\alpha_j - y)dx + (\alpha_j - x)dy), \\ u_j^\beta &= c_j^\beta (\beta_j d\mu_1^+ - d\mu_2^+) = c_j^\beta ((\beta_j - y)dx + (\beta_j - x)dy). \end{aligned}$$

Thus, evaluating $d\mathbf{H}$ on the facets of D corresponding to $x = \alpha_j$ (resp. $y = \beta_j$), we obtain that our boundary condition reduces to $A'(\alpha_j) = 2/c_j^\alpha$ (resp. $B'(\beta_j) = -2/c_j^\beta$).

7.2.2. Hyperbolic type. By (30) (specialized to $q(z) = z$) we have $g_- = \left(\frac{x+y}{x-y}\right)^2 g_+$ so that in order g_- be globally defined we need to impose (additionally to Assumption 2) $x^2 - y^2 > 0$ on D , i.e.

$$(63) \quad \alpha_1 + \beta_1 > 0.$$

Similar discussion, based on relation (43)–(44), implies in this case that the polytope Δ^+ is of the form

$$(64) \quad \begin{aligned} \Delta^+ &= \{(\mu_1^+, \mu_2^+) : L_j^\alpha(\mu_1^+, \mu_2^+) \geq 0; L_j^\beta(\mu_1^+, \mu_2^+) \geq 0, j = 1, 2\}, \\ L_j^\alpha(\mu_1^+, \mu_2^+) &= (-1)^j (-\alpha_j^2 \mu_1^+ + \mu_2^+ - \alpha_j), \\ L_j^\beta(\mu_1^+, \mu_2^+) &= (-1)^{j+1} (-\beta_j^2 \mu_1^+ + \mu_2^+ - \beta_j). \end{aligned}$$

The inward normals are then

$$(65) \quad u_j^\alpha = c_j^\alpha (\alpha_j^2, -1), \quad u_j^\beta = c_j^\beta (\beta_j^2, -1), \quad j = 1, 2,$$

for some real numbers c_j^α, c_j^β , satisfying $c_1^\alpha > 0 > c_2^\alpha$, $c_1^\beta < 0 < c_2^\beta$. The condition that $(\Delta^+, u_j^\alpha, u_j^\beta)$ is a simply connected rational Delzant polytope is again (61). We find, moreover,

$$(66) \quad \mathbf{H} = \frac{1}{(x-y)(x+y)^3} \begin{pmatrix} A(x) + B(y) & y^2 A(x) + x^2 B(y) \\ y^2 A(x) + x^2 B(y) & y^4 A(x) + x^4 B(y) \end{pmatrix}.$$

The first order boundary conditions of Proposition 9 again reduce to

- $A(z)$ and $B(z)$ are smooth functions;
- for $j \in \{1, 2\}$, $A(\alpha_j) = B(\beta_j) = 0$, $A'(\alpha_j) = 2/c_j^\alpha$ and $B'(\beta_j) = -2/c_j^\beta$;
- $A > 0$ on (α_1, α_2) and $B > 0$ on (β_1, β_2) .

7.2.3. *Elliptic type.* The metrics g_\pm are defined globally if and only if x, y satisfy $x > y$ and $(1 + xy) > 0$ (see (30)). Thus, we require in this case that

$$(67) \quad \alpha_j \beta_i > -1, \forall i, j.$$

By (48)–(49), the polytope Δ^+ is of the form

$$(68) \quad \begin{aligned} \Delta^+ &= \{(\mu_1^+, \mu_2^+) : L_j^\alpha(\mu_1^+, \mu_2^+) \geq 0; L_j^\beta(\mu_1^+, \mu_2^+) \geq 0, j = 1, 2\}, \\ L_j^\alpha(\mu_1^+, \mu_2^+) &= (-1)^j \left(\frac{(1 - \alpha_j^2)}{2} \mu_1^+ + \alpha_j \mu_2^+ + 1 \right), \\ L_j^\beta(\mu_1^+, \mu_2^+) &= (-1)^{j+1} \left(\frac{(1 - \beta_j^2)}{2} \mu_1^+ + \beta_j \mu_2^+ + 1 \right). \end{aligned}$$

The inward normals are then

$$(69) \quad u_j^\alpha = c_j^\alpha \left(\frac{(\alpha_j^2 - 1)}{2}, -\alpha_j \right), \quad u_j^\beta = c_j^\beta \left(\frac{(\beta_j^2 - 1)}{2}, -\beta_j \right), \quad j = 1, 2,$$

for some real numbers c_j^α, c_j^β , satisfying $c_1^\alpha > 0 > c_2^\alpha$, $c_1^\beta < 0 < c_2^\beta$. The condition that $(\Delta^+, u_j^\alpha, u_j^\beta)$ is a simply connected rational Delzant polytope is again (61).

Furthermore, in this case \mathbf{H} is $\frac{1}{(x-y)(1+xy)^3}$ times the matrix

$$\begin{pmatrix} 4y^2 A(x) + 4x^2 B(y) & 2y(y^2 - 1)A(x) + 2x(x^2 - 1)B(y) \\ 2y(y^2 - 1)A(x) + 2x(x^2 - 1)B(y) & (y^2 - 1)^2 A(x) + (x^2 - 1)^2 B(y) \end{pmatrix}.$$

One directly checks that

$$\begin{aligned} \mathbf{H}_{(\alpha_j, y)}(u_j^\alpha, \cdot) &\equiv 0, \quad \mathbf{H}_{(x, \beta_j)}(u_j^\beta, \cdot) \equiv 0 \\ d\mathbf{H}_{(\alpha_j, y)}(u_j^\alpha, u_j^\alpha) &= c_j^\alpha A'(\alpha_j) u_j^\alpha, \quad d\mathbf{H}_{(x, \beta_j)}(u_j^\beta, u_j^\beta) = -c_j^\beta B'(\beta_j) u_j^\beta. \end{aligned}$$

We summarize the preceding analysis as follows.

Proposition 11. *Any compact, simply connected ambikähler toric 4-orbifold for which (g_+, ω_+, J_+) is of parabolic, hyperbolic or elliptic type on M^0 , is respectively determined by the following data:*

- a simply connected, rational polytope Δ^+ defined by (59)–(60), (64)–(65), or (68)–(69), for some real numbers $\alpha_j, \beta_j, c_j^\alpha, c_j^\beta$ ($j = 1, 2$), subject to the integrality condition (61), and the inequalities

$$\begin{aligned} \beta_1 &< \beta_2 < \alpha_1 < \alpha_2, \\ c_1^\alpha &> 0 > c_2^\alpha, \quad c_1^\beta < 0 < c_2^\beta, \end{aligned}$$

as well as (63) or (67), when g_+ is of hyperbolic or elliptic type, respectively;

- two smooth functions of one variable, $A(z)$ and $B(z)$, satisfying

$$A(\alpha_j) = 0 = B(\beta_j); \quad A'(\alpha_j) = 2/c_j^\alpha; \quad B'(\beta_j) = -2/c_j^\beta, \quad j = 1, 2,$$

and $A(z) > 0$ on (α_1, α_2) and $B(z) > 0$ on (β_1, β_2) .

8. EXTREMAL AND CONFORMALLY EINSTEIN KÄHLER ORBIFOLDS

In this section we construct explicit examples of *extremal* ambikähler orbifolds. To this end, we consider that the functions $A(z)$ and $B(z)$ in Proposition 11 are polynomials of degree ≤ 4 , satisfying the relations specified in the table of section 6.3. Using the boundary conditions, each of these polynomials is determined by the data $(\alpha_j, \beta_j, c_j^\alpha, c_j^\beta)$, up to one free constant which, without loss, can be taken to be one of the respective coefficients. The three additional linear relations between the coefficients of $A(z)$ and $B(z)$ corresponding to the extremality condition thus impose one nontrivial algebraic (rational) relation between the data $(\alpha_j, \beta_j, c_j^\alpha, c_j^\beta)$ (should such polynomials $A(z), B(z)$ exist). Together with the integrality condition (61), this poses, in general, a nontrivial diophantine problem: we didn't attempt in this paper to classify all such 'extremal' data. Instead, we limit our discussion to the description of examples of compact extremal ambikähler orbifolds, thus leaving the classification issue for subsequent work—the parabolic (orthotoric) case has been resolved in [37]. We also concentrate here on ambitoric metrics of hyperbolic type (which provide different examples from those obtained in [8, 37]) though our arguments apply with obvious modifications to the other two cases too.

Our main strategy is to use a limiting argument to establish the existence of examples where the data are close to known (Bochner-flat) examples.

8.1. Weighted projective planes (revisited). We now describe the Bochner-flat extremal Kähler metrics of [12] on certain weighted projective planes as ambitoric metrics of hyperbolic type.

Take $B(z) := -(z - \beta_1)(z - \beta_2)(z - \beta_3)(z - \beta_4)$ where the β_i 's are all positive rational numbers ordered by $0 < \beta_1 < \beta_2 < \beta_3 < \beta_4$. We put $\alpha_1 = \beta_2, \alpha_2 = \beta_3$ and take $A(z) = -B(z)$. The corresponding Kähler metric g_+ in the ambitoric pair of hyperbolic type is therefore Bochner-flat (because $s_- = 0$, i.e., $W_- = 0$). In this case, we have $\alpha_1 = \beta_2$ and the discussion in section 7.2.2 goes through with the notable difference that g_+ compactifies on a toric orbifold whose rational Delzant polytope is the simplex

$$\Delta^+ = \begin{cases} L_2^\alpha = -\alpha_2^2 \mu_1^+ + \mu_2^+ - \alpha_2 \geq 0 \\ L_1^\beta = -\beta_1^2 \mu_1^+ + \mu_2^+ - \beta_1 \geq 0 \\ L_2^\beta = +\beta_2^2 \mu_1^+ - \mu_2^+ + \beta_2 \geq 0 \end{cases}$$

with inward normals

$$c_2^\alpha(\alpha_2^2, -1), \quad c_1^\beta(\beta_1^2, -1), \quad c_2^\beta(\beta_2^2, -1),$$

where

$$\begin{aligned} c_1^\beta &= \frac{2}{(\beta_1 - \beta_2)(\beta_1 - \beta_3)(\beta_1 - \beta_4)}, \\ c_2^\beta &= \frac{2}{(\beta_2 - \beta_1)(\beta_2 - \beta_3)(\beta_2 - \beta_4)}, \\ c_2^\alpha &= -\frac{2}{(\beta_4 - \beta_1)(\beta_4 - \beta_2)(\beta_4 - \beta_3)}. \end{aligned}$$

We know from [2, 8] that the simply connected toric orbifold corresponding to such a simplex is a weighted projective plane $\mathbb{C}P_{p_1, p_2, p_3}^2$, where p_1, p_2, p_3 are positive coprime integers. The detailed study of the curvature of the Bochner-flat metrics on weighted projective spaces in [19] allows us to recover the corresponding weights

from the geometric data: the minimum s_m , the maximum s_M and the average value \bar{s} (i.e. the integral divided on the total volume) of the scalar curvature s of g_+ . These values can be calculated explicitly, by using the formulae in section 6. Specifically, we get

$$(70) \quad \begin{aligned} s_m &= 6(\beta_1\beta_4 - \beta_2\beta_3); \\ s_M &= 6(\beta_3\beta_4 - \beta_1\beta_2), \\ \bar{s} &= 2\beta_4(\beta_1 + \beta_2 + \beta_3) - 2(\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1). \end{aligned}$$

From [19] one deduces immediately that the weights $p_1 < p_3 < p_3$ are determined, up to a common multiple, by

$$\begin{aligned} p_1 &= (\beta_4 - \beta_3)(\beta_1 + \beta_2) \\ p_2 &= (\beta_4 - \beta_2)(\beta_1 + \beta_3) \\ p_3 &= (\beta_4 - \beta_1)(\beta_2 + \beta_3). \end{aligned}$$

These formulae show that any weights $0 < p_1 < p_2 < p_3$ can be realized by this construction.

We notice that $s_m > 0$ (i.e. the scalar curvature s is everywhere positive) provided that β_4 is big enough.

At the other extreme, when β_4 is close to β_3 , the scalar curvature is positive at $(x, y) = (\beta_2, \beta_1), (\beta_3, \beta_1)$ and is negative at $(x, y) = (\beta_3, \beta_2)$ (so that s is zero along a line intersecting Δ^+). A particular situation which we will exploit later is when β_4 is close to β_3 as above, and $\beta_1, \beta_2, \beta_3$ satisfy $\beta_2^2 \leq \beta_1\beta_3$: in this case the scalar curvature is also positive at $(x, y) = (\beta_2, \beta_2)$.

8.2. Extremal ambitoric compactifications. We now give examples of ambitoric (orbifold) compactifications of the extremal ambitoric Kähler metrics of hyperbolic type.

We start with the polynomial $B(z) = -z^4 + b_3z^3 + b_2z^2 + b_1z + b_0$ defined in section 8.1, where $(-1)^{k+1}b_k$ is the k -th elementary symmetric function of $\beta_1, \beta_2, \beta_3, \beta_4$. For any rational numbers (α_1, α_2) such that $\alpha_2 > \alpha_1 \geq \beta_2$, we consider the polynomial $A(z) = z^4 + a_3z^3 - b_2z^2 + a_1z - b_0$, where the rational coefficients a_3, a_1 are determined from the equalities $A(\alpha_1) = A(\alpha_2) = 0$. Put $c_1^\alpha := 2/A'(\alpha_1)$, $c_2^\alpha := 2/A'(\alpha_2)$. By construction, when $\alpha_1 = \beta_2$ and $\alpha_2 = \beta_3$, we have $A(z) = -B(z)$. It follows that $c_1^\alpha > 0 > c_2^\alpha$ and $A(z) > 0$ on (α_1, α_2) for any choice of (α_1, α_2) sufficiently close to (β_2, β_3) . By Remark 5 and Proposition 11, we get countably infinitely many examples of extremal ambitoric orbifolds of hyperbolic type (parameterized by the rational numbers $\beta_1, \beta_2, \beta_3, \beta_4, \alpha_1, \alpha_2$).

8.3. Conformally Einstein Kähler orbifolds. We consider next Bach-flat ambitoric orbifolds of hyperbolic type. Pairs of polynomials of degree ≤ 4 , $A(z)$ and $B(z)$, satisfying the extremality relations of section 6.2 are parameterized with 7 unknown coefficients; there is an overall scaling factor for $A(z)$ and $B(z)$ (introducing homothetic metrics), so that such pairs can be identified with points of \mathbb{P}^6 . The Bach-flat condition introduces an additional rational hyper-surface in \mathbb{P}^6 (a quadric). If we fix the positive rational numbers α_j, β_j , the 4 end-point (linear) relations $A(\alpha_j) = 0 = B(\beta_j)$ define a $\mathbb{P}^2 \subset \mathbb{P}^6$. It intersects our quadric in a conic. So we can find pairs of polynomials with rational coefficients verifying the Bach-flat condition and vanishing at the given points. Such pairs determine rational Delzant polytopes (by putting $c_j^\alpha = 2/A'(\alpha_j)$, $c_j^\beta = -2/B'(\beta_j)$) once we have the right signs of c_j^α and c_j^β (see Remark 5). There is still the positivity condition for $A(z)$ and

$B(z)$ to be verified in order to get Bach-flat ambitoric orbifolds, for which we will use a limiting argument as in section 8.2.

Specifically, let $B(z) = -z^4 + b_3 z^3 + b_2 z^2 + b_1 z + b_0$ be the polynomial introduced in section 8.1 and consider the 2-parameter family of polynomials

$$B_{\gamma,\delta}(z) := -z^4 + \delta b_3 z^3 + b_2(\gamma, \delta) z^2 + \gamma b_1 z + b_0(\gamma, \delta),$$

where $b_2(\gamma, \delta)$ and $b_1(\gamma, \delta)$ are rational linear functions of γ and δ , defined from the equalities $B_{\gamma,\delta}(\beta_1) = B_{\gamma,\delta}(\beta_2) = 0$. Thus, by construction, $B_{1,1}(z) = B(z)$. Set

$$A_{\gamma,\delta}(z) = z^4 - \gamma b_3 z^3 - b_2(\gamma, \delta) z^2 - \delta b_1 z - b_0(\gamma, \delta).$$

Then the pair $(A_{\gamma,\delta}(z), B_{\gamma,\delta}(z))$ satisfy the Bach-flat conditions in section 6.2. Now, for any rational numbers α_1, α_2 with $\beta_2 \leq \alpha_1$, we set $A_{\gamma,\delta}(\alpha_1) = A_{\gamma,\delta}(\alpha_2) = 0$, which are rational linear conditions that determine (γ, δ) .

Thus, we constructed a family of polynomials (whose coefficients are rational functions of $\beta_1, \beta_2, \beta_3, \beta_4, \alpha_1, \alpha_2$) satisfying the Bach-flat conditions. If we take $\alpha_1 = \beta_2$ and $\alpha_2 = \beta_3$ we get the pair $(-B(z), B(z))$ of section 8.1. By continuity, for any choice of (α_1, α_2) sufficiently closed to (β_2, β_3) , the corresponding polynomials will satisfy the positivity conditions and $c_j^\alpha = 2/A'(\alpha_j)$, $c_j^\beta = -2/B'(\beta_j)$ will satisfy $c_1^\alpha > 0 > c_2^\alpha$, $c_1^\beta < 0 < c_2^\beta$. We thus get countably-infinitely many examples of Bach-flat ambitoric orbifolds.

Notice that if the scalar curvature s_+ of the corresponding Bach-flat Kähler metric g_+^K is positive on M (equivalently, on Δ^+), then $g := \frac{1}{s_+^K} g_+^K$ is an Einstein, hermitian metric on M . Positivity of s_+ can be obtained by a limiting argument as above, if the β_j 's are chosen so that the scalar curvature of the corresponding Bochner-flat metric is positive (see section 8.1).

Example 1. A specific pair of polynomials defining a Bach-flat ambitoric orbifold (with $\beta_1 = 0, \beta_2 = 1, \alpha_1 = 2, \alpha_2 = 3$) is given by

$$A(z) = -z^4 + 4z^3 - z^2 - 6z, \quad B(z) = z^4 - 6z^3 + z^2 + 4z.$$

8.4. Complete Einstein metrics. This is similar to the construction of complete Einstein selfdual examples in [12]. Suppose that (M, g_+, ω_+) is an ambitoric Bach-flat orbifold such that the scalar curvature s_+ (which is an affine function of (μ_1^+, μ_2^+)) is negative on a triangle $T \subset \Delta^+$. Clearly, $\mu^{-1}(T)$ is a compact orbifold with boundary (equal to the zero locus of s_+). By the Delzant construction, this orbifold is covered by a compact manifold with boundary N : Indeed, let C be the cone in \mathbb{R}^2 defined by the two facets of Δ_+ which bound T and Λ be the lattice generated by the normals to the these facets; we thus can identify (C, Λ) as the image by the momentum map of a (standard) toric \mathbb{C}^2 ; thus, the pre-image N of $T \subset C$, is the closure of a bounded domain in \mathbb{C}^2 . Since $ds_+ \neq 0$ on ∂N , the lift of $g = \frac{1}{s_+^K} g_+^K$ to $N \setminus \partial N$ defines a conformally compact, Einstein hermitian metric of negative scalar curvature; such a metric is automatically complete, see e.g. [3].

Examples of such ambitoric Bach-flat orbifolds can be found by the limiting argument in section 8.3, once we have taken $\beta_1, \beta_2, \beta_3, \beta_4$ such that the scalar curvature of the corresponding Bochner-flat metric is positive at $(\beta_2, \beta_1), (\beta_3, \beta_1), (\beta_2, \beta_2)$, and negative at (β_3, β_2) (see section 8.1).

9. RELATION TO K-STABILITY AND NONEXISTENCE RESULTS

We restrict attention to the case that $\alpha_j, \beta_j, c_j^\alpha, c_j^\beta$ are all rational numbers (see Remark 5). Then the corresponding polytope Δ^+ defines a toric orbifold and one

can ask how our constructions are related to the stability notions introduced in [23, 43, 45]. Referring to these works for the relevant definitions and motivation, we briefly recall below the material relevant to the toric orbifolds we consider.

9.1. The extremal vector field of a toric orbifold. Let (M, J, g, ω) be a compact Kähler orbifold invariant under the action of a maximal compact subgroup G of the reduced group of automorphisms $H_0(M, J)$ of (M, J) . (By a result of Calabi, any extremal Kähler metric is invariant under such a G .) Following [26], the *extremal vector field* of (M, g, J, ω, G) is the Killing vector field whose Killing potential is the L_2 -projection of the scalar curvature s_g of g to the space of Killing potentials (with respect to g) of elements of the Lie algebra \mathfrak{g} . Futaki and Mabuchi [26] showed that this definition is independent of the choice of a G -invariant Kähler metric within the given Kähler class $\Omega = [\omega]$ on (M, J) . Since the extremal vector field is necessarily in the centre of \mathfrak{g} , it can be equally defined if we take instead of G a maximal torus \mathbb{T} in $H_0(M, J)$. This remark is relevant to the case of toric orbifolds, where compatible Kähler metrics are all invariant under the (maximal) m -torus \mathbb{T} , and belong to the same Kähler class when the corresponding complex structures are identified via an equivariant diffeomorphism (see e.g. [38, 23]). Thus, in the notation of section 7.1, the extremal vector field of a toric orbifold (M, J, g, ω) is the hamiltonian vector field corresponding to the affine function $-\langle A, \mu \rangle - B$ on \mathfrak{t}^* , where $A \in \mathfrak{t}$ and $B \in \mathbb{R}$ are introduced by the following equations:

$$\begin{cases} \sum_s \alpha_s A_s + \alpha B + 2\beta &= 0, \\ \sum_s \alpha_{rs} A_s + \alpha_r B + 2\beta_r &= 0, \end{cases}$$

with

$$\begin{aligned} \alpha &= \int_{\Delta} dv, & \alpha_r &= \int_{\Delta} \mu_r dv, & \alpha_{rs} &= \int_{\Delta} \mu_r \mu_s dv, \\ \beta &= \frac{1}{2} \int_{\Delta} s_g dv, & \beta_r &= \frac{1}{2} \int_{\Delta} s_g \mu_r dv, \end{aligned}$$

where $dv = d\mu_1 \wedge \dots \wedge d\mu_m$ is the volume form of \mathfrak{t}^* . In the sequel, we will identify the extremal vector field with the corresponding element $-A \in \mathfrak{t}$.

In order to express β and β_r only in terms of Δ , we use the formula for the scalar curvature of a compatible Kähler metric, found in [1]:

$$(71) \quad s_g = - \sum_{r,s} \frac{\partial^2}{\partial \mu_r \partial \mu_s} (H_{rs}),$$

where $\mathbf{H} = (H_{rs})$ is the $S^2 \mathfrak{t}^*$ -valued function corresponding to g (see section 7.1). By applying the divergence theorem (with inward normals) and the boundary conditions for \mathbf{H} (see Proposition 9), we obtain

$$\begin{aligned} \beta &= \frac{1}{2} \int_{\Delta} s_g dv = \int_{\partial \Delta} d\nu, \\ \beta_r &= \frac{1}{2} \int_{\Delta} s_g \mu_r dv = \int_{\partial \Delta} \mu_r d\nu \end{aligned}$$

Here $d\nu$ is the $(m-1)$ -form on $\partial \Delta$ with $u_i \wedge d\nu = -dv$ on the facet with normal u_i .

This computation does not use the fact that \mathbf{H} is positive definite on the faces of Δ . It follows that the extremal vector field can be computed from any smooth \mathbf{H} which satisfies the boundary conditions of Proposition 9.

9.2. K-energy and the relative Futaki invariant. If we parameterize compatible Kähler metrics g by their symplectic potentials $U(\mu)$ (with $\mathbf{H} = (\text{Hess } U)^{-1}$ satisfying the boundary conditions of Proposition 9), then the relative K-energy \mathcal{E} (of Mabuchi, Guan, and Simanca) on this space is introduced by the functional equation (see [23, 45])

$$\begin{aligned} (d\mathcal{E})_g(\dot{U}) &= \int_{\Delta} \left(\langle A, \mu \rangle + B - \sum_{r,s} \frac{\partial^2}{\partial \mu_r \partial \mu_s} (H_{rs}) \right) \dot{U}(\mu) dv \\ &= 2 \int_{\partial \Delta} \dot{U}(\mu) d\nu + \int_{\Delta} \left(\langle A, \mu \rangle + B \right) \dot{U}(\mu) dv - \int_{\Delta} \langle \mathbf{H}, \text{Hess } \dot{U}(\mu) \rangle dv, \end{aligned}$$

where we have applied twice the divergence theorem by using the boundary conditions for \mathbf{H} given by Proposition 9.

Still following [23, 45], in the toric setting we introduce the *relative Futaki invariant* as the functional

$$(72) \quad \mathcal{F}(U) := \int_{\partial \Delta} U(\mu) d\nu + \frac{1}{2} \int_{\Delta} \left(\langle A, \mu \rangle + B \right) U(\mu) dv$$

acting on the space of symplectic potentials U , so that, using the fact that the derivative of $\log \det V$ is $\text{tr } V^{-1} dV$, we obtain the formula

$$\mathcal{E}(U) = 2\mathcal{F}(U) - \int_{\Delta} (\log \det \text{Hess } U(\mu)) dv,$$

where, as Donaldson shows, the convergence of the integrals (in $(-\infty, \infty]$) follow from the convexity of U .

Note that $\mathcal{F}(f)$ is well-defined for any continuous function $f(\mu)$ on Δ , and the relationship between the relative Futaki invariant and the derivative of the relative K-energy shows that if $f(\mu)$ is an affine linear function, then $\mathcal{F}(f) = 0$.

9.3. K-stability for toric orbifolds. Following the work of Donaldson [23] and Székelyhidi [43], the notion of (relative) *K-stability with respect to toric degenerations* of a smooth toric variety (M, ω) turns out to be the key to understanding the existence of compatible extremal Kähler metric. This notion was originally introduced in [23, 43] as an algebro-geometric construction in the framework of the GIT, but is now fully understood in terms of the corresponding Delzant polytope Δ and its associated relative Futaki functional (72), see e.g. [43, Prop. 4.1.3]. This reformulation has the advantage to generalize for toric orbifolds (not only manifolds), and even for polytopes which are not necessarily rational Delzant [25]. Combining terminology of [40] and [43], we give the following definition:

Definition 13. Let (M, ω) be a $2m$ -dimensional compact toric orbifold with Delzant polytope Δ and relative Futaki invariant \mathcal{F} . Suppose that f is a continuous piecewise-linear (PL) concave function on Δ , i.e., f is the minimum of a finite collection of affine linear functions. Then, (M, ω) is *analytically relative K-polystable* with respect to toric degenerations if $\mathcal{F}(f) \geq 0$ for any PL concave function f with equality if and only if f is an affine function.

The main conjecture from [23] is that a compact toric orbifold admits a compatible extremal Kähler metric if and only if (M, ω) is analytically relative K-polystable with respect to toric degenerations.

In one direction, the conjecture has been already established: by a result in [46], if (M, ω) admits an extremal Kähler metric then it is analytically relative K-polystable with respect to toric degenerations.

In the other direction, a recent result of Donaldson [25] shows that for polygons with zero extremal vector field, the K-polystability implies existence of a CSC metric.

The general extremal case remains open, which motivates us to look at the problem in the special cases we study here.

9.4. K-stability for ambitoric polytopes.

Theorem 4. *Let (M, ω) be a toric orbifold with Delzant polytope Δ corresponding to the data $(\alpha_j, \beta_j, c_j^\alpha, c_j^\beta)$ as in Proposition 11. Suppose that there are polynomials of degree ≤ 4 , $A(z)$ and $B(z)$, which verify the corresponding first-order boundary conditions at α_j and β_j (but not necessarily the positivity conditions) and the corresponding extremality relation given in section 6.2. Then, the following conditions are equivalent*

- (i) (M, ω) admits an extremal Kähler metric;
- (ii) (M, ω) is analytically relative K-polystable with respect to toric degenerations;
- (iii) $A(z) > 0$ on (α_1, α_2) and $B(z) > 0$ on (β_1, β_2) .

In particular, if (M, ω) admits an extremal Kähler metric, this metric must be ambitoric.

Proof. For given A, B , let \mathbf{H} be the corresponding matrix with entries $H_{rs} = g_+^K(K_r, K_s)$. Since $A(z)$ and $B(z)$ are polynomials satisfying the corresponding extremal relations, at least at the points where $A(x)$ and $B(y)$ are nonzero we have a genuine (perhaps indefinite) extremal Kähler metric. Using the expression (71) of the scalar curvature in momentum coordinates and the fact that \mathbf{H} verifies the first-order boundary conditions on $\partial\Delta$, we obtain

$$(73) \quad \sum_{r,s} \frac{\partial^2}{\partial \mu_r \partial \mu_s} (H_{rs}) = \langle A, \mu \rangle + B,$$

where $-A$ is the extremal vector field of Δ . Substituting back in (72), and integrating by parts, we get

$$\begin{aligned} \mathcal{F}(U) &= \int_{\partial\Delta} U(\mu) d\nu + \frac{1}{2} \int_{\Delta} \left(\sum_{r,s} \frac{\partial^2}{\partial \mu_r \partial \mu_s} (H_{rs}) \right) U(\mu) d\nu \\ &= \frac{1}{2} \int_{\Delta} \text{trace}(\mathbf{H} \text{ Hess } U) d\nu. \end{aligned}$$

This formula makes sense for functions U which are at least twice differentiable, but it can be also used to calculate the action (in distributional sense) of \mathcal{F} on simple PL functions: let f be a simple concave PL function with crease on the line $\{\langle u_f, \mu \rangle = -c_f\}$ and S_f be the intersection of this line with Δ . Then

$$\mathcal{F}(f) = \int_{S_f} \mathbf{H}(u_f, u_f) d\nu_f,$$

where ν_f is the positive measure on S_f introduced by the equality $u_f \wedge d\nu_f = d\nu$. This formula holds for *any* matrix which satisfies (73) and the first-order boundary conditions on $\partial\Delta$. In particular, if (M, ω) admits an extremal (toric) metric, we can take the corresponding \mathbf{H} (which will be positive definite): it follows then that $\mathcal{F}(f) > 0$ for any simple concave PL function meeting Δ .

For the polytopes we consider there is a special choice of line segments as follows: for any $x_0 \in (\alpha_1, \alpha_2)$ consider the line segment in (x, y) coordinates $\{(x_0, y) : y \in (\beta_1, \beta_2)\}$. It transforms in momentum coordinates to a line segment S_{x_0} in the

interior of Δ . Let u_{x_0} be a normal of S_{x_0} . It is straightforward to check that in any of the three cases $\mathbf{H}_{(x_0,y)}(u_{x_0}, u_{x_0})$ is positive multiple of $A(x_0)$. Thus, if (M, ω) is analytically relative K-polystable with respect to toric degenerations, then $A(x_0)$ must be positive for any $x_0 \in (\alpha_1, \alpha_2)$; the argument for $B(z)$ is similar.

We thus showed that either of the conditions (i) and (ii) implies (iii). However, if $A(z)$ and $B(z)$ are positive, our construction produces an ambitoric extremal Kähler metric on (M, ω) , so that (iii) implies (i). The implication (i) \Rightarrow (ii) follows from the main result of [46, Theorem 1.3].

Our final assertion follows from the uniqueness of the extremal toric Kähler metrics, modulo automorphisms, established in [28]. \square

Remark 6. In the light of [23] and its extension to orbifolds in [41], when the rational Delzant polytope Δ has rational vertices with respect to the dual lattice, one can also consider a weaker version of *algebraic* relative K-polystability with respect to toric degenerations, by requiring that $\mathcal{F}(f) \geq 0$ for any *rational* PL continuous concave function f with equality if and only if f is an affine function. Presumably, this is the condition corresponding to the algebro-geometric notion of K-stability for the corresponding orbifold. A key observation in [23] is that in the case of a rational polygon with vanishing extremal vector field, the algebraic relative K-polystability with respect to toric degenerations is equivalent to the analytic one. This phenomenon is well demonstrated on our extremal examples too: if $\alpha_j, \beta_j, c_j^\alpha, c_j^\beta$ are all rational numbers (so that the vertices of Δ are rational with respect to the dual lattice) and if $\mathcal{F} > 0$ on rational PL concave functions which are not affine on Δ , we then conclude as in the proof of Theorem 4 that $A(z)$ must be positive at any *rational* point in (α_1, α_2) . It follows that $A(z) \geq 0$ on (α_1, α_2) with (possibly) a repeated irrational root in this interval. As the α_i 's and c_j^α 's are rational, by the first order boundary conditions $A(z)$ is a (multiple of) degree 4 polynomial with rational coefficients with two simple (rational) roots α_1 and α_2 . In particular, any double root of A (if any) must be rational too, showing that $A(z)$ must be strictly positive on (α_1, α_2) . Similarly, $B(z) > 0$ on (β_1, β_2) .

Remark 7. If we make the construction in section 8.2 starting with a polynomial $B(z)$ which vanishes on (β_1, β_2) , we obtain many examples of toric orbifolds which do not admit extremal Kähler metric at all.

APPENDIX A. KILLING TENSORS AND AMBITORIC CONFORMAL METRICS

The material in this appendix is related to recent work of W. Jelonek [32, 33, 34] and some well-known results in General Relativity, see [18] and [35]. To provide a different slant, we take a conformal viewpoint (cf. [14, 16, 27, 42]) and make explicit the connection with Pontecorvo's description [39] of hermitian structures which are conformally Kähler. We specialize the analysis to ambitoric structures.

A.1. Conformal Killing objects. Let (M, c) be a conformal manifold. Among the conformally invariant linear differential operators on M , there is a family which are overdetermined of finite type, sometimes known as twistor or Penrose operators; their kernels are variously called twistors, tractors, or other names in special cases. Among the examples where the operator is first order are the equations for twistor forms (also known as conformal Killing forms) and conformal Killing tensors, both of which include conformal vector fields as a special case. There is also a second order equation for Einstein metrics in the conformal class. Apart from the obvious presence of (conformal) Killing vector fields and Einstein metrics, conformal Killing 2-tensors and twistor 2-forms are very relevant to the present work.

Let $S_0^k TM$ denote the bundle of symmetric $(0, k)$ -tensors \mathcal{S}_0 which are tracefree with respect to c in the sense that $\sum_i \mathcal{S}_0(\varepsilon_i, \varepsilon_i, \cdot) = 0$ for any conformal coframe ε_i . In particular, for $k = 2$, $\mathcal{S}_0 \in S_0^2 TM$ may be identified with $\sigma_0 \in L^2 \otimes \text{Sym}_0(TM)$ via $\alpha \circ \sigma_0(X) = \mathcal{S}_0(\alpha, c(X, \cdot))$ for any 1-form α and vector field X . Here $\text{Sym}_0(TM)$ is the bundle of tracefree endomorphisms of TM which are symmetric with respect to c ; thus σ_0 satisfies $c(\sigma_0(X), Y) = c(X, \sigma_0(Y))$ and hence defines a (weighted) $(2, 0)$ -tensor S_0 in $L^4 \otimes S_0^2 T^*M$, another isomorph of $S_0^2 TM$ (in the presence of c).

A *conformal Killing 2-tensor* is a section \mathcal{S}_0 of $S_0^2 TM$ such that the section $\text{sym}_0 D\mathcal{S}_0$ of $L^{-2} \otimes S_0^3 TM$ is identically zero, where D is any Weyl connection (such as the Levi-Civita connection of any metric in the conformal class) and sym_0 denotes orthogonal projection onto $L^{-2} \otimes S_0^3 TM$ inside $T^*M \otimes S^2 TM \cong L^{-2} \otimes TM \otimes S^2 TM$. Equivalently $\text{sym } D\mathcal{S}_0 = \text{sym}(\chi \otimes c)$ for some vector field χ . Taking a trace, we find that $(n+2)\chi = 2\delta^D \mathcal{S}_0$, where $\delta^D \mathcal{S}_0$ denotes $\text{tr}_c D\mathcal{S}_0$, which may be computed, using a conformal frame e_i with dual coframe ε_i , as $\sum_i D_{e_i} \mathcal{S}_0(\varepsilon_i, \cdot)$. Thus \mathcal{S}_0 is conformal Killing iff

$$(74) \quad \text{sym } D\mathcal{S}_0 = \frac{2}{n+2} \text{sym}(c \otimes \delta^D \mathcal{S}_0),$$

This is independent of the choice of Weyl connection D . On the open set where \mathcal{S}_0 is nondegenerate, there is a unique such D with $\delta^D \mathcal{S}_0 = 0$, and hence a nondegenerate \mathcal{S}_0 is conformal Killing if and only if there is a Weyl connection D with $\text{sym } D\mathcal{S}_0 = 0$.

A *conformal Killing 2-form* is a section ϕ of $L^3 \otimes \wedge^2 T^*M$ such that $\pi(D\phi) = 0$ (for any Weyl connection D) where π is the projection orthogonal to $L^3 \otimes \wedge^3 T^*M$ and $L \otimes T^*M$ in $T^*M \otimes L^3 \otimes \wedge^2 T^*M$. It is often more convenient to identify ϕ with a section Φ of $L \otimes \mathfrak{so}(TM)$ via $\phi(X, Y) = c(\Phi(X), Y)$, where $\mathfrak{so}(TM)$ denotes the bundle of skew-symmetric endomorphisms of TM with respect to c .

A.2. Conformal Killing tensors and complex structures. In four dimensions a conformal Killing 2-form splits into selfdual and antiselfdual parts Φ_\pm , which are sections of $L \otimes \mathfrak{so}_\pm(TM) \cong L^3 \otimes \wedge_\pm^2 T^*M$. Following Pontecorvo [39], nonvanishing conformal Killing 2-forms Φ_+ and Φ_- describe oppositely oriented Kähler metrics in the conformal class, by writing $\Phi_\pm = \ell_\pm J_\pm$, where ℓ_\pm are sections of L and J_\pm are oppositely oriented complex structures: the Kähler metrics are then $g_\pm = \ell_\pm^{-2} c$. Conversely if J_\pm are Kähler with Kähler metrics $g_\pm = \ell_\pm^{-2} c$ and D^\pm denote the Levi-Civita connections of g_\pm then $D^\pm(\ell_\pm J_\pm) = 0$ so $\Phi_\pm = \ell_\pm J_\pm$ are conformal Killing 2-forms.

The tensor product of sections Φ_+ and Φ_- of $L \otimes \mathfrak{so}_+(TM)$ and $L \otimes \mathfrak{so}_-(TM)$ defines a section $\Phi_+ \Phi_-$: as a section of $L^2 \otimes \text{Sym}_0(TM)$, this is simply the composite $(\Phi_+ \circ \Phi_- = \Phi_- \circ \Phi_+)$; as a section of $L^4 \otimes S_0^2 T^*M$ it satisfies $(\Phi_+ \Phi_-)(X, Y) = c(\Phi_+(X), \Phi_-(Y))$.

When $\Phi_\pm = \ell_\pm J_\pm$ are nonvanishing, $\Phi_+ \Phi_- = \ell_+ \ell_- J_+ J_-$ is a symmetric endomorphism with two rank 2 eigenspaces at each point. Conversely if σ_0 is such a symmetric endomorphism, we may write $\sigma_0 = \ell^2 J_+ J_-$ for uniquely determined almost complex structures J_\pm up to overall sign, and a positive section ℓ of L .

Proposition 12. *A nonvanishing section $\sigma_0 = \ell^2 J_+ J_-$ of $L^2 \otimes \text{Sym}_0(TM)$ (as above) is associated to a conformal Killing 2-tensor \mathcal{S}_0 if and only if J_\pm are integrable complex structures which are “Kähler on average” with length scale ℓ , in the sense that if D^\pm denote the canonical Weyl connections of J_\pm , then the connection $\frac{1}{2}(D^+ + D^-)$ preserves the length scale ℓ (i.e., $D^+ \ell + D^- \ell = 0$).*

If these equivalent conditions hold, then also $\text{sym } D\mathcal{S}_0 = 0$.

(With respect to an arbitrary metric g in the conformal class, the “Kähler on average” condition means that the Lee forms θ_\pm^g satisfy $d(\theta_g^+ + \theta_g^-) = 0$. In the case that J_+ and J_- both define conformally Kähler metrics g_\pm , the metric $g_0 = \ell^{-2}c$ is the barycentric metric with $g_0 = f g_+ = f^{-1}g_-$ for some function p .)

Proof. Let D, D^+, D^- be Weyl connections with $D = \frac{1}{2}(D^+ + D^-)$ in the affine space of Weyl connections. (Thus the induced connections on L are related by $D = D^+ + \theta = D^- - \theta$ for some 1-form θ .) Straightforward calculation shows that

$$D\sigma_0 = D(\ell^2) \otimes J_+ \circ J_- + \ell^2(D^+ J_+ \circ J_- + J_+ \circ D^- J_-) + R$$

where R is an expression (involving θ) whose symmetrization vanishes (once converted into a trilinear form using c). If J_\pm are integrable and Kähler on average, then taking D^\pm to be the canonical Weyl connections and ℓ the preferred length scale, $\ell^2 J_+ J_-$ is thus associated to a conformal Killing tensor S_0 with $\text{sym } DS_0 = 0$.

For the converse, it is convenient (for familiarity of computation) to work with the associated $(2, 0)$ -tensor S_0 with $S_0(X, Y) = \ell^2 c(J_+ J_- X, Y)$. Since S_0 is nondegenerate, and associated to a conformal Killing tensor, we can let $D = D^+ = D^-$ be the unique Weyl connection with $\text{sym } DS_0 = 0$: note that $\text{sym}: L^4 \otimes T^*M \otimes S^2 T^*M \rightarrow L^4 \otimes S^3 T^*M$ here becomes the natural symmetrization map. Thus

$$\sum_{X,Y,Z} D_X(\ell^2) c(J_+ \circ J_- Y, Z) = \sum_{X,Y,Z} \ell^2 \left(c((D_X J_+) J_- Y, Z) + c(J_+ (D_X J_-) Y, Z) \right),$$

where the sum is over cyclic permutations of the arguments. If X, Y, Z belong to a common eigenspace of S_0 then the right hand side is zero—this follows because, for instance, $c((D_X J_\pm) J_\pm Y, Z)$ is skew in Y, Z whereas the cyclic sum of the two terms is totally symmetric.

It follows that $D\ell = 0$, hence the right hand side is identically zero in X, Y, Z . Additionally $c(D_X J_\pm \cdot, \cdot)$ is J_\pm -anti-invariant. Thus these 2-forms vanish when their arguments have opposite types $((1, 0)$ and $(0, 1)$) with respect to the corresponding complex structure. Now suppose for example that Z_1 and Z_2 have type $(1, 0)$ with respect to J_+ , but opposite types with respect to J_- (J_+ and J_- are simultaneously diagonalizable on $TM \otimes \mathbb{C}$). Then by substituting first $X = Y = Z_1, Z = Z_2$ into

$$\sum_{X,Y,Z} c((D_X J_+) J_- Y, Z) = \sum_{X,Y,Z} c((D_X J_-) Y, J_+ Z),$$

and then $X = Y = Z_2, Z = Z_1$, we readily obtain

$$c((D_{Z_1} J_+) Z_1, Z_2) = 0 = c((D_{Z_2} J_+) Z_1, Z_2).$$

Thus $D_{J_+ X} J_+ = J_+ D_X J_+$ for all X and J_+ is integrable. Similarly, we conclude J_- is integrable. \square

Since D is the Levi-Civita connection D^g of the “barycentric” metric $g = \ell^{-2}c$, it follows that $S_0 = g(J_+ J_- \cdot, \cdot)$ is a *Killing tensor* with respect to g , i.e., satisfies $\text{sym } D^g S_0 = 0$ if and only if J_+ and J_- are integrable and Kähler on average, with barycentric metric g . More generally, we can use this result to characterize, for any metric g in the conformal class and any functions f, h , the case that

$$(75) \quad S(\cdot, \cdot) = f g(\cdot, \cdot) + h g(J_+ J_- \cdot, \cdot),$$

is a Killing tensor with respect to g . If θ_\pm are the Lee forms of (g, J^\pm) , i.e., $D^\pm = D^g \pm \theta_\pm$, then we obtain the following more general corollary.

Corollary 2. $S = f g + h g(J_+ J_- \cdot, \cdot)$, with h nonvanishing, is a Killing tensor with respect to g if and only if:

$$(76) \quad J_+ \text{ and } J_- \text{ are both integrable;}$$

$$(77) \quad \theta_+ + \theta_- = -\frac{dh}{h};$$

$$(78) \quad J_+ df = J_- dh.$$

(Obviously if h is identically zero, S is a Killing tensor if and only if f is constant.)

A.3. Conformal Killing tensors and the Ricci tensor. The tracefree part $Ric_0^g = Ric^g - \frac{1}{n} Scal^g g$ of the Ricci tensor of a compatible metric $g = \mu_g^{-2} c$ on a conformal n -manifold (M, c) defines a tracefree symmetric $(0, 2)$ -tensor $\mathcal{S}_0^g(\alpha, \beta) = Ric_0^g(\alpha^\sharp, \beta^\sharp)$ (where for $\alpha \in T^*M$, $g(\alpha^\sharp, \cdot) = \alpha$), where the corresponding section of $L^4 \otimes S_0^2 T^*M$ is $S_0 = \mu_g^4 Ric_0^g$.

The differential Bianchi identity implies that $0 = \delta^g(Ric^g - \frac{1}{2} Scal^g g) = \delta^g Ric_0^g - \frac{n-2}{2n} dScal^g$. Hence the following are equivalent:

- S_0 is a conformal Killing tensor;
- $\text{sym } D^g S_0 = \frac{n-2}{n(n+2)} \text{sym}(g^{-1} \otimes dScal^g)$;
- $Ric^g - \frac{2}{n+2} Scal^g g$ is a Killing tensor with respect to g ;
- $D_X^g Ric^g(X, X) = \frac{2}{n+2} dScal^g(X)g(X, X)$ for all vector fields X .

Riemannian manifolds (M, g) satisfying these conditions were introduced by A. Gray as \mathcal{AC}^\perp -manifolds [29]. Relevant for this paper is the case $n = 4$ and the assumption that Ric^g has two rank 2 eigendistributions, which has been extensively studied by W. Jelonek [33, 34].

Supposing that g is not Einstein, Corollary 2 implies, as shown by Jelonek, that

$$Ric^g - \frac{1}{3} Scal^g g = f g + h g(J_+ J_- \cdot, \cdot)$$

is Killing with respect to g iff (76)–(78) are satisfied. Since J_\pm are both integrable, Jelonek refers to such manifolds as *bihermitian Gray surfaces*. It follows from [4] that both (g, J_+) and (g, J_-) are conformally Kähler, so that in the context of the present paper, a better terminology would be *ambikähler Gray surfaces*.

However, the key feature of such metrics is that the Ricci tensor is J_\pm -invariant: as long as J_\pm are conformally Kähler, Proposition 6 applies to show that the manifold is either ambitoric or of Calabi type; it is not necessary that the J_\pm -invariant Killing tensor constructed in the proof is equal to the Ricci tensor of g .

Jelonek focuses on the case that the ambihermitian structure has Calabi type. This is justified by the global arguments he employs. In the ambitoric case, there are strong constraints, even locally.

A.4. Killing tensors and hamiltonian 2-forms. The notion of *hamiltonian 2-forms* on a Kähler manifold (M, g, J, ω) has been introduced and extensively studied in [6, 7]. According to [7], a J -invariant 2-form ϕ is hamiltonian if it verifies the equation

$$(79) \quad D_X \phi = \frac{1}{2} (d\sigma \wedge JX^\flat - Jd\sigma \wedge X^\flat),$$

for any vector field X , where $X^\flat = g(X)$ and $\sigma = \text{tr}_\omega \phi = g(\phi, \omega)$ is the trace of ϕ with respect to ω . An essentially equivalent (but not precisely the same) definition

was given in the four dimensional case in [6], by requiring that a J -invariant 2-form φ is closed and its primitive part φ_0 satisfies

$$(80) \quad D_X \varphi_0 = -\frac{1}{2} d\sigma(X)\omega + \frac{1}{2} (d\sigma \wedge JX^\flat - Jd\sigma \wedge X^\flat),$$

for some smooth function σ . Note that, in order to be closed, φ is necessarily of the form $\frac{3}{2}\sigma\omega + \varphi_0$.

The relation between the two definitions is straightforward: $\varphi = \frac{3}{2}\sigma\omega + \varphi_0$ is closed and verifies (80) if and only if $\phi = \varphi_0 + \frac{1}{2}\sigma\omega$ satisfies (79).

Specializing Corollary 2 to the case when the metric g is Kähler with respect to $J = J_+$ allows us to identify J -invariant symmetric Killing tensors with hamiltonian 2-forms as follows:

Proposition 13. *Let S be a symmetric J -invariant tensor on a Kähler surface (M, g, J, ω) , and $\psi(\cdot, \cdot) = S(J\cdot, \cdot)$ be the associated J -invariant 2-form. Then S is Killing if and only if $\phi = \psi - (\text{tr}_\omega \psi)\omega$ is a hamiltonian 2-form (i.e. verifies (79)).*

Proof. As observed in [7, p. 407], ϕ satisfies (79) if and only if $\varphi = \phi + (\text{tr}_\omega \phi)\omega$ is a closed 2-form and $\psi = \phi - (\text{tr}_\omega \phi)\omega$ is the 2-form associated to a J -invariant Killing tensor (this is true in any complex dimension $m > 1$).

Noting that the 2-forms φ and ψ are related by $\varphi = \psi - \frac{2\text{tr}_\omega \psi}{m-1}\omega$, we claim that in complex dimension $m = 2$, the 2-form $\varphi = \phi - 2(\text{tr}_\omega \psi)\omega$ is automatically closed, provided that ψ is the 2-form associated to a J -invariant Killing tensor S . Indeed, under the Kähler assumption the conditions (76)–(77) specialize as

$$(81) \quad J_- \text{ is integrable,}$$

$$(82) \quad \theta_- = -\frac{dh}{h},$$

It follows that $(g_- = h^{-2}g, J_-, \omega_- = g_-(J_-\cdot, \cdot))$ defines a Kähler metric. From (75) we have

$$(83) \quad \psi = f\omega_+ + h^3\omega_-,$$

where $\omega_+ = g(J_+\cdot, \cdot)$ denotes the Kähler form of (g, J_+) . In particular, the trace of φ with respect to ω_+ is equal to $2f$ while the condition (78) and the fact that ω_- is closed imply that $\varphi = \psi - 4f\omega_+ = -3f\omega_+ + h^3\omega_-$ is closed too. \square

A.5. Killing tensors associated to ambitoric structures. We have seen in the previous sections that there is a link between Killing tensors and ambihermitian structures. We now make this link more explicit in the case of ambitoric metrics.

In the ambitoric situation, the barycentric metric g_0 (see section 5) satisfies $\theta_+^0 + \theta_-^0 = 0$. It then follows from Corollary 2 that the (tracefree) symmetric bilinear form $g_0(I\cdot, \cdot)$ (with $I = J_+ \circ J_-$) is Killing with respect to g_0 . More generally, let g be any (K_1, K_2) -invariant riemannian metric in the ambitoric conformal class c , so that g can be written as $g = hg_0$ for some positive function $h(x, y)$, where x, y are the coordinates introduced in section 5. Then $\theta_+^g + \theta_-^g = -d\log h$. From Corollary 2 again, the symmetric bilinear form $S_0(\cdot, \cdot) = hg_0(I\cdot, \cdot)$ is conformal Killing. Moreover, by condition (78) in Proposition 2, it can be completed into a Killing symmetric bilinear form $S = fg + S_0$ if and only if the 1-form $dh \circ I$ is closed. Since $I dx = -dx$ and $I dy = dy$, $dh \circ I$ is closed if and only if $h_x dx - h_y dy$ is closed, if and only if $h_{xy} = 0$; the general solution is $h(x, y) = F(x) - G(y)$, for some functions F, G . Note that the coefficient $f(x, y)$ is determined by $df = -Idh = F'(x)dx + G'(y)dy$ (see (78)), so we can take without loss $f(x, y) = F(x) + G(y)$. Thus, S is Killing, with eigenvalues (with respect to g) equal to $2F(x)$ and $2G(y)$.

A similar argument shows that any conformal metric of the form $g = f(z)g_0$, where g_0 is the barycentric metric of an ambikähler pair of Calabi type and z is the momentum coordinate introduced in section 4.2, admits a nontrivial symmetric Killing tensor of the form $S(\cdot, \cdot) = f(z)g(\cdot, \cdot) + f(z)g(I\cdot, \cdot)$ (and hence with eigenvalues $(2f(z), 0)$).

It follows that there are infinitely many conformal metrics in a given ambitoric conformal class, which admit nontrivial symmetric Killing tensors.

There are considerably fewer such metrics with diagonal Ricci tensor however. If the ambitoric structure is induced by a metric g with diagonal Ricci tensor, then we must have $g = (\mu_-^{(p)} + c)^{-2}g_- = (\mu_+^{[w]})^{-2}g_+$ for some constants $p = (p_0, p_1, p_2)$ with $\langle q, p \rangle = 0$, a constant of integration c , and constants $w = (w_0, w_1, w_2)$. Since $(x - y)^2g_- = q(x, y)^2g_+$, this is satisfied iff $c = 0$ and $w = p$ (hence also $\langle q, w \rangle = 0$). Thus $g = h(x, y)g_0$ where

$$h(x, y) = \frac{(x - y)q(x, y)}{p(x, y)^2}.$$

In order for g to admit a nontrivial symmetric Killing tensor, we must have $h_{xy} = 0$. A calculation shows that this happens iff $Q(p) = 0$ (i.e., $p(z)$ has repeated roots). Since p is orthogonal to q , this can only happen if $Q(q) \geq 0$ and there are generically ($Q(q) > 0$) just two solutions for p , which coincide if $Q(q) = 0$.

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